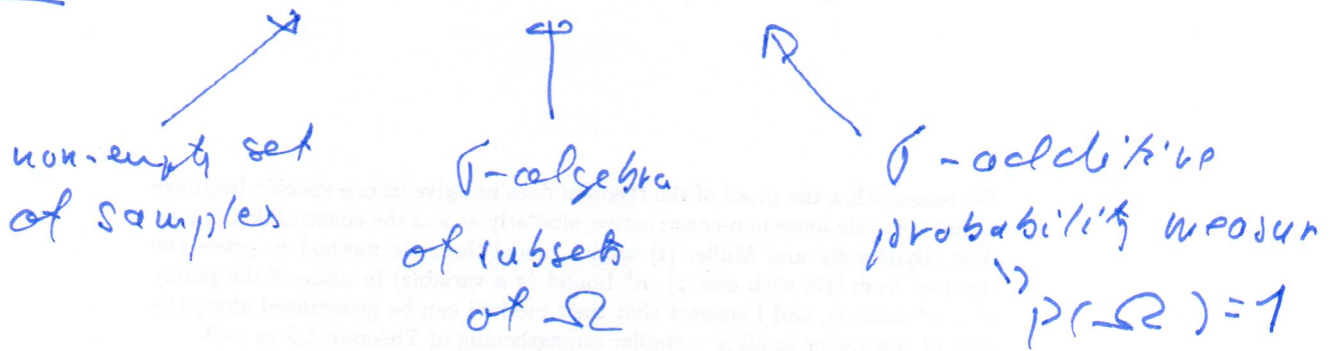


Set-up: (Ω, \mathcal{A}, P)



$[P=0] :=$ the ideal in \mathcal{A} of sets A , $P(A) = 0$

$B := \mathcal{A} / [P=0] : \text{Boolean algebra}$

Lemma (p. 96 in Scott's paper)

B is complete: $\forall U \subseteq B \exists \sup U$.

Proof:

Claim 1: B is σ -algebra and $\mu: B \rightarrow [0,1]$ defined by

$$\mu(A/[P=0]) := P(A)$$

is a σ -additive measure on B that

is strictly positive (i.e. $\forall b \in B, b \neq 0, \mu(b) > 0$).

Prf-claim 1: Use that $[P=0]$ is closed under countable unions (as P is σ -additive) \square_1

Claim 2: \mathcal{B} has ccc (= countable chain condition) $\uparrow \downarrow$ ccf

Every antichain $I \subseteq \mathcal{B}$ is countable.

$\uparrow \downarrow$ ccf

$\forall b_1, b_2 \in I, b_1 \neq b_2 \rightarrow b_1 \wedge b_2 = \emptyset_{\mathcal{B}}$

Prf-claim 2: By the strict positivity of μ there can be $\leq k$ elements of I with μ -measure in the interval $(\frac{1}{k+1}, \frac{1}{k}]$, for all $k = 1, 2, \dots$ \square_2

Claim 3: For $U \subseteq \mathcal{B}, U \neq \emptyset$, denote \tilde{U} the ideal generated by U ($b \in \tilde{U}$ iff $b \leq a_1 \vee \dots \vee a_n$, some $a_i \in U$).

Then the upper bounds of U and \tilde{U} are the same: $\forall b \in \mathcal{B}: b \geq U \Leftrightarrow b \geq \tilde{U}$.

\square_3

Claim 4 : Any ideal \tilde{U} has the same upper bounds as any maximal antichain $I \subseteq \tilde{U}$.
 (Follows by Zorn's lemma.)

Prf-claim 4: Clear $b \geq \tilde{U} \Rightarrow b \geq I$.

If for some b : $b \geq I$ but $b \not\geq \tilde{U}$, say $b \not\geq a$ for $a \in \tilde{U}$, then $a \wedge b$ ($:= a \wedge \bar{b}$) $\neq \emptyset$ and $I \cup \{a \wedge b\}$ is also antichain, contradicting the maximality of I . \square

Claim 5: $\forall U \subseteq B \exists V \subseteq U$, V countable & (U and V have the same upper bounds)

Prf-claim 5: U has the same upp. bounds as \tilde{U} by Claim 3, which has the same upper bounds as an max. antichain $I \subseteq \tilde{U}$ by Claim 4, and I is countable by Claim 2. Define:

V : some countable $\subseteq U$ s.t. $I \subseteq \tilde{V}$ (the ideal generated by)

exists, because each element of \tilde{U} is majorized by the union of finitely many elmts of U . \square

To conclude the proof of the lemma
note that V provided by Claim 4 has the
sup V by Claim 1 (i.e. by the δ -additivity)

q.e.d.