Scott67 – p. 96

Set-up: $(\Omega, \mathcal{A}, P)$

non-empty set of samples

σ-algebra of subsets of Ω

σ-additive probability measure $\leadsto P(\Omega) = 1$

$[P=0] :=$ the ideal in $\mathcal{A}$ of sets X, $P(X) = 0$

$\mathcal{B} := \mathcal{A}/_{[P=0]}$: Boolean algebra

Lemma (p. 96 in Scott's paper)

$\mathcal{B}$ is complete: $\forall U \subseteq \mathcal{B}\ \exists \sup U$.

Proof:

Claim 1: $\mathcal{B}$ is σ-algebra and $\mu : \mathcal{B} \to [0,1]$ defined by

$$\mu\left(A/_{[P=0]}\right) := P(A)$$

is a σ-additive measure on $\mathcal{B}$ that is strictly positive (i.e. for $b \in \mathcal{B}\setminus\{0\}$, $\mu(b) > 0$).

Prf-claim 1: Use that $[P=0]$ is closed under countable unions (as P is σ-additive)

$\square_1$

Claim 2: $\mathbb{B}$ has ccc (= countable chain condition)

$\Updownarrow$ df

Every antichain $I \subseteq \mathbb{B}$ is countable.

$\Updownarrow$ df

$\forall b_1, b_2 \in I, b_1 \neq b_2 \rightarrow b_1 \wedge b_2 = 0_{\mathbb{B}}$

Prf-claim 2: By the strict positivity of μ there can be $\leq k$ elements of I with μ-measure in the interval $\left[\frac{1}{k+1}, \frac{1}{k}\right]$, for all $k = 1, 2, \ldots$.

$\square_2$

Claim 3: For $U \subseteq \mathbb{B}$, $U \neq \emptyset$, denote $\tilde{U}$ the ideal generated by U ($b \in \tilde{U}$ iff $b \leq a_1 \vee \ldots \vee a_\ell$, some $a_i \in U$).

Then the upper bounds of U and $\tilde{U}$ are the same: $\forall b \in \mathbb{B}$: $b \geq U \iff b \geq \tilde{U}$.

$\square_3$

Claim 4 : Any ideal $\tilde{\mathcal{U}}$ has the same upper bounds as any maximal antichain $I \subseteq \tilde{\mathcal{U}}$.
(Exists by Zorn's lemma.)

Prf-claim 4: Clear $b \geq \tilde{\mathcal{U}} \Rightarrow b \geq I$.
If for some b: $b \geq I$ but $b \not\geq \tilde{\mathcal{U}}$, say $b \not\geq a$ for $a \in \tilde{\mathcal{U}}$, then $a - b \; (:= a \cap \bar{b}) \neq \emptyset$ and $I \cup \{a - b\}$ is also antichain, contradicting the maximality of I. $\square_4$

Claim 5: $\forall \mathcal{U} \subseteq \mathcal{B} \; \exists \mathcal{V} \subseteq \mathcal{U}$, $\mathcal{V}$ countable & ($\mathcal{U}$ and $\mathcal{V}$ have the same upper bounds).

Prf-claim 5: $\mathcal{U}$ has the same upp. bounds as $\tilde{\mathcal{U}}$ by Claim 3, which has the same upper bounds as any max. antichain $I \subseteq \tilde{\mathcal{U}}$ by Claim 4, and I is countable by Claim 2. Define:
$\mathcal{V}$: some countable $\subseteq \mathcal{U}$ s.t. $I \subseteq \tilde{\mathcal{V}}$ (the ideal generated by)
exists, because each element of $\tilde{\mathcal{U}}$ is majorized by the union of finitely many elmt's of $\mathcal{U}$. $\square_5$

To conclude the proof of the lemma note that V provided by Claim 4 has the Sup V by Claim 1 (i.e. by the δ-additivity).

q.e.d.