# A saturation property of structures obtained by forcing with a compact family of random variables


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**Abstract**

A method for constructing Boolean-valued models of some fragments of arithmetic was developed in [3], with the intended applications in bounded arithmetic and proof complexity. Such a model is formed by a family of random variables defined on a pseudo-finite sample space. We show that under a fairly natural condition on the family (called compactness in [3]) the resulting structure has a property that is naturally interpreted as saturation for existential types. We also give an example showing that this cannot be extended to universal types.


Let $K$ be a Boolean-valued $L$-structure. That is, each sentence $A$ in the language $L(K)$, $L$ augmented by constants for all elements of $K$, is assigned a truth-value $[\![A]\!]$ in a complete Boolean algebra $\mathcal{B}$. These values commute with propositional connectives (after Boole[1]) and satisfy

$$[\![\exists x A(x)]\!] \ = \ \bigvee_{u\in K} [\![A(u)]\!] \quad \text{and} \quad [\![\forall x A(x)]\!] \ = \ \bigwedge_{u\in K} [\![A(u)]\!]$$

(after Rasiowa-Sikorski[5]).

Let $\mathbf{p}$ be a set of formulas in variables $x = x_1, \dots, x_n$. In the classical case the set $\mathbf{p}$ is an **$n$-type** over a structure if it is **finitely satisfied**, i.e.

$$\bigwedge_{\Phi\subseteq_f \mathbf{p}} \exists x \bigwedge_{A\in\Phi} A(x)$$


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where $\Phi$ runs over finite subsets of $\mathbf{p}$, and it is **realized** in the structure if

$$\exists x \bigwedge_{A \in \mathbf{p}} A(x)$$

holds there. In the context of Boolean-valued structures this is naturally transcribed as the following condition on truth values:

$$(\mathsf{Satur}) \qquad \bigwedge_{\Phi \subseteq_f \mathbf{p}} [\![\exists x \bigwedge_{A \in \Phi} A(x)]\!] \quad \leq \quad \bigwedge_{A \in \mathbf{p}} [\![A(u)]\!]$$

for some $u \in K^n$; we shall say that such $u$ **realizes (Satur) for p** in the structure. Note that in that case (Satur) is actually an equality as the left-hand side always majorizes the right-hand side.

In this paper we show that a certain class of Boolean-valued structures constructed via forcing with random variables (recalled in Sections 1 and 2) is saturated in this sense for sets of existential formulas. This is done first for the special case of sets $\mathbf{p}$ consisting of open formulas in Section 3 to display clearly the idea, and for the existential case in Section 4. In Section 5 we give an example of a structure from the same class that is not saturated for sets of universal formulas. The paper is concluded by a brief explanation in Section 6 how is open saturation of the structures considered potentially relevant to proof complexity.

Background on model theory can be found in [4], further material (and details) on forcing with random variables in [3].

# 1 Forcing with random variables set-up

In this section we shall briefly recall the construction of Boolean valued structures by forcing with random variables from [3]. The intended target structures are models of arithmetic with a special emphasis on bounded arithmetic. This is motivated by a close relation of bounded arithmetic to proof complexity but we shall not review this topic here (an interested reader can consult [2, 3]).

The structures are built from a family of random variables on a pseudo-finite sample space. Let $\mathcal{M}$ be a non-standard $\aleph_1$-saturated model of true arithmetic in some language $L$ containing the language of Peano arithmetic and having a canonical interpretation in the standard model $\mathbf{N}$. In [3] we took a language having symbols for all relations and functions on $\mathbf{N}$ but here it is natural to consider countable $L$. In fact, we shall assume that $L$ is definable in Peano arithmetic (this includes finite and recursive languages).

Let $\Omega \in \mathcal{M}$ be an infinite set; as it is an element of the model it is $\mathcal{M}$-finite. Let $F \subseteq \mathcal{M}$ be any family of functions $\alpha : \Omega \to \mathcal{M}$. We call elements of $F$ random variables. It is not assumed that $F$ is definable in the ambient model $\mathcal{M}$.

Let $\mathcal{A}$ be the Boolean algebra of $\mathcal{M}$-definable subsets of $\Omega$ and let $\mathcal{B}$ be its quotient by the ideal $\mathcal{I}$ of sets of an infinitesimal counting measure. Using the idea of Loeb's measure, the $\aleph_1$-saturation of $\mathcal{M}$ and some measure theory it was shown in [3] that $\mathcal{B}$ is a complete Boolean algebra.

The counting measure on $\mathcal{A}$ induces **a strict measure** $\mu$ (in the ordinary sense with values in $\mathbf{R}$) on $\mathcal{B}$. The measure defines a metric on $\mathcal{B}$: the distance of two elements is the measure of their symmetric difference.

For any $k$-ary function symbol $f$ from $L$ and any $\alpha_1, \ldots, \alpha_k \in F$ define the function $f(\alpha_1, \ldots, \alpha_k) : \Omega \to \mathcal{M}$ by

$$f(\alpha_1, \ldots, \alpha_k)(\omega) \; := \; f(\alpha_1(\omega), \ldots, \alpha_k(\omega)), \quad \text{for } \omega \in \Omega \; .$$

If this function is also always in $F$ we say that $F$ is **$L$-closed**.

Any $L$-closed family $F$ is the universe of a Boolean-valued $L$-structure $K(F)$ with $L(F)$-sentences having their truth values in $\mathcal{B}$ defined as follows. Every atomic $L(F)$-sentence $A$ is naturally assigned a set $\langle\langle A \rangle\rangle$ from $\mathcal{A}$ consisting of those samples $\omega \in \Omega$ for which $A$ is true in $\mathcal{M}$. The image of $\langle\langle A \rangle\rangle$ in $\mathcal{B}$, the quotient $\langle\langle A \rangle\rangle / \mathcal{I}$, is denoted $[\![A]\!]$. Following Boole [1] and Rasiowa-Sikorski [5] this determines the truth value $[\![A]\!] \in \mathcal{B}$ for any $L(F)$-sentence $A$: $[\![\ldots]\!]$ commutes with Boolean connectives and

$$[\![\exists x A(x)]\!] \; := \; \bigvee_{\alpha \in F} [\![A(\alpha)]\!] \text{ and } [\![\forall x A(x)]\!] \; := \; \bigwedge_{\alpha \in F} [\![A(\alpha)]\!] \; .$$

It holds that all logically valid sentences get the maximal truth value $1_{\mathcal{B}}$. We say that a sentence is **valid in** $\mathcal{M}$ if its truth value is $1_{\mathcal{B}}$.

There are various generalizations of this basic set-up considered in [3]. For example, the random variables from the family $F$ can be only partially defined on the sample space $\Omega$ (as long as their regions of undefinability have infinitesimal counting measures) or the sample space may be equipped with some other probability distribution than the uniform one.

We shall use one immediate consequence of the $\aleph_1$-saturation of $\mathcal{M}$ and so we formulate it as a lemma.

**Lemma 1.1** *Let $\{a_k\}_{k \in \mathbf{N}}$ be any sequence of elements of $\mathcal{M}$. Then there is an element $a^* \in \mathcal{M}$ that codes a sequence $\{a^*_i\}_{i \le t}$ of some non-standard length $t \in \mathcal{M} \setminus \mathbf{N}$ such that $a^*_k = a_k$ for all $k \in \mathbf{N}$.*

Any such element $a^*$ is called **a non-standard extension** of $\{a_k\}_{k \in \mathbf{N}}$. We shall skip in future the $^*$ in the notation and denote a non-standard extension simply as $\{a_i\}_{i \le t}$.

# 2 Compact families and witnessing of quantifiers

In this section we shall recall the concept of a compact family $F$ from [3, Chpt.3] and some properties of $K(F)$ it implies.

**Definition 2.1** *Let $F \subseteq \mathcal{M}$ be a family.*

1. *$F$ is* **closed under definitions by cases by open $L$-formulas** *iff whenever $\alpha, \beta \in F$ and $B(x)$ is an open $L(F)$-formula with free variable $x$ then there is $\gamma \in F$ such that:*

$$\gamma(\omega) = \begin{cases} \alpha(\omega) & \textit{if } B(\alpha(\omega)) \textit{ holds} \\ \beta(\omega) & \textit{otherwise.} \end{cases}$$

2. *$F$ is* **compact** *iff there exists an $L$-formula $H(x,y)$ such that for*

$$F_a := \{b \in \mathcal{M} \mid \mathcal{M} \models H(a,b)\}$$

*the following two properties hold:*

- $\bigcap_{k \in \mathbf{N}} F_k = F$.
- $F_k \supseteq F_{k+1}$*, for all* $k \in \mathbf{N}$.

Recall that the Overspill is the principle, a simple consequence of induction, that if a property definable in $\mathcal{M}$ holds for all standard numbers, it must hold actually for all elements up to some non-standard element of $\mathcal{M}$ (cf. [3, Appendix]). The primary intended use of compact families is to allow the following type of reasoning.

Assume $\{\alpha_k\}_{k \in \mathbf{N}}$ is an arbitrary sequence of elements of $F$ and that $\{\alpha_i\}_{i \leq t}$ is its non-standard extension. Note that the conditions posed on sets $F_k$ in the definition of compactness imply that for all standard $k$ it holds that

- $\forall j \leq k \ \alpha_j \in F_k$, and
- $\forall j < k \ F_j \supseteq F_{j+1}$.

By the Overspill in $\mathcal{M}$ this must hold also for some non-standard $s \leq t$. In particular, all $\alpha_j$ from the non-standard extension with $j \leq s$ are in $F_s \subseteq F$, and hence in $F$ too.

**Theorem 2.2** *[3, Thm.3.5.2]*

*Let $F$ be an $L$-closed family that is closed under definition by cases by open $L$-formulas and compact. Let $A$ be an $L(F)$-sentence of the form*

$$\exists x_1 \forall y_1 \ldots \exists x_k \forall y_k B(x_1, y_1, \ldots, x_k, y_k)$$

*with $B$ open.*

*Then there are $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in F$ such that for all $i = 1, \ldots, k$:*

$$[\![\forall y_i \exists x_{i+1} \forall y_{i+1} \ldots \exists x_k \forall y_k B(\alpha_1, \beta_1, \ldots, \alpha_i, y_i, x_{i+1}, y_{i+1}, \ldots, x_k, y_k)]\!] = [\![A]\!]$$

*and*

$$[\![\exists x_{i+1} \forall y_{i+1} \ldots \exists x_k \forall y_k B(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, x_{i+1}, y_{i+1}, \ldots, x_k, y_k)]\!] = [\![A]\!] \ .$$

## 3 Saturation for sets of open formulas

In this section we prove a special case of saturation when all formulas in the set $\mathbf{p}$ are open. We will use the following immediate corollary of Theorem 2.2. To simplify the notation we consider, here as well as in the next section, sets of formulas in one free variable; the general case would be done in the same way.

**Corollary 3.1** *Let $F$ be an $L$-closed family that is closed under definition by cases by open $L$-formulas and compact. Let $A(x)$ be an open $L(F)$-formula with one free variable $x$.*

*Then there is $\alpha \in F$ such that:*

$$[\![\exists x A(x)]\!] = [\![A(\alpha)]\!] \ = \ \langle\!\langle A(\alpha)\rangle\!\rangle/\mathcal{I} \ .$$

**Proof :**

The first equality follows from Theorem 2.2 and the second one follows by the definition of $[\![A]\!]$ as $A$ is open and taking the quotient by $\mathcal{I}$ commutes with Boolean connectives.

**q.e.d.**

In proving the next theorem we could restrict to the special case when the left-hand side of (Satur) has value $1_{\mathcal{B}}$ by taking a suitable quotient of $\mathcal{B}$. However, this would be done at the expense of having to show that the inequality for the resulting new Boolean-valued structure could be pulled back to the original one. We thus prefer not to make this simplification.

**Theorem 3.2** *Let $L$ be a language definable in Peano arithmetic. Let $F$ be an $L$-closed family that is closed under definition by cases by open $L$-formulas and compact.*

*Assume that $\mathbf{p}$ is a countable set of open $L(F)$-formulas in one free variable $x$.*

*Then there is an element $\alpha \in F$ that realizes the saturation inequality* (Satur) *for the set $\mathbf{p}$ in $K(F)$.*

**Proof :**

Let $A'_1(x), A'_2(x), \ldots$ enumerate $\mathbf{p}$ and define

$$A_k(x) \ := \ \bigwedge_{i \leq k} A'_i(x) \ , \quad \text{for } k \geq 1 \ .$$

Then we have the following

**Claim 1:** *All implications*

$$A_{k+1}(x) \rightarrow A_k(x)$$

*are logically valid and if an element $\alpha \in F$ satisfies the following inequality*

$$\bigwedge_k [\![\exists x A_k(x)]\!] \ \leq \ \bigwedge_k [\![A_k(\alpha)]\!]$$

*then $\alpha$ realizes the* (Satur) *inequality for* $\mathbf{p}$ *in* $K(F)$*.*

Note that the inequality is actually an equality in that case.

**Claim 2:** *Assume* $\alpha \in F$, $U \subseteq \Omega \wedge U \in \mathcal{M}$ *and it holds*

$$\langle\langle A_k(\alpha) \rangle\rangle \supseteq U$$

*for all* $k \in \mathbf{N}$, *and*

$$\mu([\![\exists x A_k(x)]\!]) \searrow \mu(U/\mathcal{I})$$

*as standard* $k \to \infty$*.*

*Then $\alpha$ satisfies the inequality from Claim 1 and hence realizes (*Satur*) for* $\mathbf{p}$ *in* $K(F)$*.*

The arrow $\searrow$ means that the sequences of reals on the left-hand side is non-increasing and its limit is the right-hand side. The claim follows as

$$[\![\exists x A_k(x)]\!]) \;\geq\; [\![A_k(\alpha)]\!]) \;\geq\; U/\mathcal{I} \;.$$

Now we are going to show that some $\alpha$ and $U$ satisfying the hypothesis of Claim 2 do exist. This will prove the theorem.

By Corollary 3.1 there are $\alpha_k \in F$ such that

$$[\![\exists x A_k(x)]\!] \;=\; [\![A_k(\alpha_k)]\!] \;.$$

Define $U_k := \langle\langle A_k(\alpha_k) \rangle\rangle$, so

$$[\![\exists x A_k(x)]\!] \;=\; U_k/\mathcal{I}$$

and as $A_\ell$ logically implies $A_k$ for $l \geq k$, also

$$U_k \;\supseteq\; U_\ell \;, \quad \text{for } \ell \geq k \;.$$

Consider the sequence $\{A_k, \alpha_k, U_k\}_{k \in \mathbf{N}}$ and let

$$\{A_i, \alpha_i, U_i\}_{i \leq t} \;, \quad t \text{ non-standard}$$

be its non-standard extension provided by Lemma 2.1.

By the compactness of $F$ there is a definable family $\{F_a\}_a$ of sets such that $F = \bigcap_{k \in \mathbf{N}} F_k$. Consider the following property, definable in $\mathcal{M}$ (the definability of $L$ is used here), of an element $i \leq t$. It is the conjunction of seven conditions:

1. $A_i$ is an open $L(F_i)$-formula.

2. $A_i \to A_j$ is logically valid for all $j \leq i$.

3. $\langle\langle A_i(\alpha_i) \rangle\rangle = U_i$.

4. $\Omega \supseteq U_1 \supseteq \ldots \supseteq U_i$.

5. $\frac{|U_j|}{|\Omega|} - \frac{|U_i|}{|\Omega|} < 1/j$ for all $j \leq i$.

6. $F_1 \supseteq \ldots \supseteq F_i$.

7. $\alpha_j \in F_i$ for all $j \leq i$.

All seven conditions are valid for all standard $i$ perhaps with the exception of 5: but taking a suitable subsequence of the original sequence $\{A_k, \alpha_k, U_k\}_{k\in\mathbf{N}}$ arranges this condition too. The first two items are included because in the argument below we need to talk in $\mathcal{M}$ also about the satisfiability relation for formulas $A_i$ with a non-standard index. By the Overspill then the property must be true for all $i \leq s$ up to some non-standard $s \leq t$. We want to show that for such an $s$, $\alpha_s$ and $U_s$ satisfy the hypothesis of Claim 2.

By 6 and 7 $\alpha_s \in F_s \subseteq F$ and by 1, 2 and 3

$$\langle\langle A_k(\alpha_s)\rangle\rangle \ \supseteq \ \langle\langle A_s(\alpha_s)\rangle\rangle \ = \ U_s$$

for all $k \in \mathbf{N}$. It remains to note that

$$\mu([\![\exists x(A_k(x)]\!]) \ \searrow \ \mu(U_s/\mathcal{I}) \ .$$

follows by 3, 4 and 5.

**q.e.d.**

In the argument we have used that for an open formula $A(x)$ the set $\langle\langle A(\alpha)\rangle\rangle$ is definable from $\alpha$ and satisfies $[\![A(\alpha)]\!] = \langle\langle A(\alpha)\rangle\rangle/\mathcal{I}$. This is not true for general formulas but for for existential formulas one could add witnesses (in the sense of Corollary 3.1) for the values $[\![A_k(\alpha_k)]\!]$ to the data in $\{A_k, \alpha_k, U_k\}_{k\in\mathbf{N}}$ and run an analogous argument.

Assuming little bit more about the structure however, we can derive the existential case directly from Theorem 3.2.

# 4 The existential case

In this section we note that Theorem 3.2 implies the statement also for sets of existential formulas as long as the underlying structure admits a pairing function. This is always the case for structures of interest in [3] as they are models of various bounded arithmetics.

**Definition 4.1** *Let $F$ be an $L$-closed family. We say that $K(F)$* **has pairing** *if $L$ contains symbols $\pi_1(x), \pi_2(x)$ for two unary functions and a symbol $\langle x, y\rangle$ for a binary function, and the universal closures of the following three formulas are valid in $K(F)$:*

- $\langle \pi_1(z), \pi_2(z)\rangle = z$.
- $\pi_1(\langle x, y\rangle) = x$.

- $\pi_2(\langle x, y\rangle) = y$.

**Theorem 4.2** *Let $L$ be a language definable in Peano arithmetic. Let $F$ be an $L$-closed family that is closed under definition by cases by open $L$-formulas and compact. Assume that $K(F)$ has pairing.*

*Let $\mathbf{p}$ be a countable set of existential $L(F)$-formulas in one free variable $x$. Then there is an element $\alpha \in F$ that realizes the saturation inequality* (Satur) *for the set $\mathbf{p}$ in $K(F)$.*

**Proof :**

Using the pairing to replace several existential quantifiers by one we may assume without a loss of generality that the formulas in $\mathbf{p}$ have the form $A(x) = \exists y B(x, y)$, with $B$ open. Enumerate $\mathbf{p}$ as $A_k(x) = \exists y B_k(x, y)$, $k \in \mathbf{N}$, and define open formulas

$$C_k(z) \ := \ B_k(\pi_1(z), \pi_1(\pi_2^{(k)}(z)))$$

where $\pi_2^{(k)}$ abbreviates $k$-times iterated $\pi_2$. The following claim implies that if $\alpha \in F$ realizes (Satur) for $\{C_k \mid k \in \mathbf{N}\}$ then $\pi_1(\alpha)$ realizes it for $\mathbf{p}$.

**Claim:** *For any $k \in \mathbf{N}$ it holds that*

$$[\![\exists x \bigwedge_{i \leq k} A_i(x)]\!] \ = \ [\![\exists z \bigwedge_{i \leq k} C_i(z)]\!] \ .$$

Clearly the left-hand side majorizes the right-hand side. For the opposite direction apply Corollary 3.1 to get $\gamma, \beta_1, \ldots, \beta_k \in F$ such that

$$[\![\exists x \bigwedge_{i \leq k} A_i(x)]\!] \ = \ \bigwedge_{i \leq k} [\![B_i(\gamma, \beta_i)]\!]$$

and define

$$\alpha \ := \ \langle \gamma, \langle \beta_1, \langle \beta_2, \ldots, \langle \beta_k, \gamma \rangle \ldots \rangle$$

(the second $\gamma$ could be replaced by any element of $F$.) It is easy to see that the substitution $z := \alpha$ gives to the right-hand side in the Claim a value that equals to $[\![\exists x \bigwedge_{i \leq k} A_i(x)]\!]$.

**q.e.d.**

# 5 The failure of the universal case

In this section we show that Theorem 4.2 cannot be generally strengthened to sets of universal formulas. Take for $L$ the language of Peano arithmetic together with the inequality sign $\leq$ and with some function symbols for pairing and its projections, and having also a unary function symbol $|x|$ for the bit-length of number $x$.

In $\mathcal{M}$ we shall identify numbers with the binary strings consisting of their bits. Let the sample space $\Omega$ be simply $\{0,1\}^n$ for some non-standard $n \in \mathcal{M}$ and let the family $F$ consist of all functions on $\Omega$ computed by circuits with $n$ inputs and arbitrarily many outputs but of the size bounded above by all terms $2^{n^{1/k}}$, for all $k \in \mathbf{N}$. In other words, these are functions computed in sub-exponential non-uniform time. It is easy to see that $F$ is compact: take for $F_k$ the functions on $\Omega$ computed by circuits of size bounded above by $2^{n^{1/k}}$.

Let $id_\Omega$ be the identity function on $\Omega$. Consider the following universal $L$-formulas:

$$A_k(x) \ := \ |x|^k \leq |id_\Omega| \wedge \forall y(|y| \neq x) \ .$$

(The expression $|x|^k$ is just an abbreviation for the term $|x| \cdot \ldots \cdot |x|$, $|x|$ occurring $k$-times.)

The value of $|id_\Omega|$ is on all samples $n$. Take for $x$ any constant function $\alpha$ outputting a fixed string of bit-length $n^{1/k}$ and of value $2^{n^{1/k}}$. Then clearly $[\![|\alpha|^k \leq |id_\Omega|]\!] = 1_{\mathcal{B}}$ but also $[\![\forall y(|y| \neq \alpha)]\!] = 1_{\mathcal{B}}$. This is because no function from $F$ can output a string of length $2^{n^{1/k}}$. This implies that the left-hand side of (Satur) for formulas $A_k(x)$ has the truth value $1_{\mathcal{B}}$.

On the other hand, assume that $\alpha$ makes all sentences $|\alpha|^k \leq |id_\Omega|$ valid. Hence the sets $U_k \subseteq \Omega$ defined by

$$\omega \in U_k \ \text{ iff } \ |\alpha(\omega)| \leq n^{1/k}$$

have all counting measures infinitesimally close to 1. The sequence of $U_k$ thus satisfies

$$U_1 \supseteq \ldots \supseteq U_k$$

and

$$1 \ - \ \frac{|U_i|}{|\Omega|} \ < \ 1/k \ , \quad \text{for all } i \leq k \ .$$

Taking its non-standard extension $\{U_i\}_{i \leq t}$ and applying the Overspill analogously as before yields a non-standard $s \leq t$ for which the counting measure of $U_s = \langle\langle |\alpha|^s \leq |id_\Omega| \rangle\rangle$ is infinitesimally close to 1, i.e. $[\![|\alpha|^s \leq |id_\Omega|]\!] = 1_{\mathcal{B}}$ holds.

But then the value of $\alpha$ on each sample from $U_s$ is at most $2^{n^{1/s}}$ and so there is a $\beta \in F$ that outputs on each sample $\omega \in U_s$ a string of bit-length $\alpha(\omega) \leq 2^{n^{1/s}}$. For such $\beta$ however, $[\![|\beta| \neq \alpha]\!] = 0_{\mathcal{B}}$. This argument proves the following statement.

**Theorem 5.1** *There is a finite language $L$ and an $L$-closed family $F$ that is closed under definition by cases by open $L$-formulas, compact and such that $K(F)$ has pairing, but for which there is a countable set of universal $L(F)$-formulas in one free variable $x$ for which no element of $K(F)$ realizes the saturation inequality* (Satur).

## 6 A concluding remark

In the abstract we have alluded to intended applications of forcing with random variables in proof complexity. To illustrate - in rather abstract terms - how open saturation can be useful consider the following situation.

Let $T$ be a theory in a countable language $L$ and let $\varphi$ be an $L$-definable 3CNF propositional formula $\varphi$. In a typical example $T$ may be an extension of a bounded arithmetic theory by an open diagram of some $L$-structure. The task would be to construct a model of $T$ containing a truth assignment satisfying $\varphi$. That is, the model should satisfy a sentence of the form

$$\exists x \forall y A(x, y)$$

where $A(x, y)$ is an open formula formalizing that the assignment $x$ satisfies the $y$-th 3-clause of $\varphi$.

If such a model can be found with $\alpha$ satisfying $\forall y A(\alpha, y)$, it often actually suffices to work further only with a substructure of the model generated by $\alpha$. But in order for this substructure to satisfy $\forall y A(\alpha, y)$ it is not necessary that the original model satisfies $\forall y A(\alpha, y)$; it suffices that $\alpha$ realizes in the model the open type consisting of countably many formulas

$$A(x, t(x))$$

with $t(x)$ ranging over all (countably many) terms with the only free variable $x$.

Open saturation allows to simplify the task to construct such a model and to consider only how to realize in it finite subsets of the type.

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