Embedding logics into product logic

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Abstract

We construct a faithful interpretation of Łukasiewicz's logic in the product logic (both propositional and predicate). Using known facts it follows that the product predicate logic is not recursively axiomatizable.

We prove a completeness theorem for the product logic extended by a unary connective $\triangle$ of Baaz [1]. We show that Gödel's logic is a sublogic of this extended product logic.

We also prove NP-completeness of the set of propositional formulas satisfiable in product logic (resp. in Gödel's logic).

1 Introduction

We shall be concerned with many-valued logics in this paper; in particular, in Łukasiewicz's logic Ł, Gödel's logic G and product logic P. Our aim is to obtain information about complexity of these logics in terms of recursive theory (in the case of predicate logic) or in terms of computational complexity theory (in the case of propositional logic). Scarpellini [13] and Mundici [9] provide such information for Łukasiewicz's logic. Hence we shall concentrate on the other two logics.

All three logics have the same language. The propositional connectives are $\overline{0}, \overline{1}, \&, \rightarrow$. The truth-values are drawn from the unit interval $[0,1]$. The values of $\overline{0}, \overline{1}$ are always 0 and 1 respectively. The three logics we consider compute the value of & by a t-norm and the value of $\rightarrow$ by the corresponding residuum. In particular, in Łukasiewicz's logic

$$x \;\&_L\; y = max(0, x+y-1)$$

and

$$x \rightarrow_L y = \begin{cases} 1 & \text{if } x \leq y \\ 1-x+y & \text{otherwise} \end{cases}$$

in Gödel's logic

$$x \;\&_G\; y = min(x,y)$$

and

$$x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

and in product logic

$$x \;\&_P\; y = x * y$$

and

$$x \rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

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The negation is defined as

$$\neg_\circ x = x \rightarrow_\circ \overline{0}$$

Minimum ($\wedge$) and maximum ($\vee$) can be defined in all three logics as $x \wedge y = x \;\&_\circ\; (x \rightarrow_\circ y)$ and $x \vee y = ((x \rightarrow_\circ y) \rightarrow_\circ y) \wedge ((y \rightarrow_\circ x) \rightarrow_\circ x)$ respectively.

An evaluation is a mapping of the set of propositional variables to the unit interval [0,1]. Every evaluation e can be extended using particular truth functions to the evaluation $e_\circ$ of all formulas (e.g. the truth value of a formula $\varphi \& \psi$ in Łukasiewicz's logic will be $e_L(\varphi \& \psi) = e_L(\varphi) \;\&_L\; e_L(\psi)$ etc.) using the truth tables of the particular logic $\circ$.

In the propositional case we ask about the existence of a complete axiomatization of the set of 1-tautologies. These are formulas that obtain the truth value 1 by each evaluation. In all three cases a complete axiomatization exists. On the other hand, if we introduce models of multi-valued predicate calculi and ask again about a complete axiomatization of 1-tautologies we get the positive answer for the Gödel's logic (see [14]), and negative (due to [13]) for the Łukasiewicz. In the following we will see how to embed Łukasiewicz's logic into product and use the fact that it is not recursively axiomatizable to prove the recursive non-axiomatizability of product predicate calculus.

We then study an extension of product logic that contains G, and complexity of propositional satisfiability in P and G. We prove NP-completeness of the propositional satisfiability in P and G.

2 Embedding Łukasiewicz's logic into product logic

In this section we will show that Łukasiewicz's t-norm $x \;\&_L\; y = max(0, x+y-1)$ can be isomorphically transformed to (restricted) product on $[a, 1]$ for arbitrary fixed $0 < a < 1$. This fact is also a direct consequence of the result of [8]. Then we will present a translation of formulas such that φ is 1-tautology of Łukasiewicz's logic if and only if its particular translation is 1-tautology of product logic. We apply this isomorphism also to models of corresponding predicate calculi and as a conclusion we get, via Scarpellini result that the set of 1-tautologies of Łukasiewicz's logic is not recursively enumerable, that the same holds for the set of 1-tautologies of product predicate calculus.

Lemma 1 For each $0 < a < 1$, [0,1] with Łukasiewicz's conjunction $x \;\&_L\; y = max(0, x+y-1)$ is isomorphic to $[a, 1]$ with restricted product $x \&_{P^a} y = max(a, x * y)$.

Proof The isomorphism is $f_a(x) = a^{1-x}$ from which $f_a^{-1}(y) = 1 - \log_a y$. We can see that $f_a(x) \&_{P^a} f_a(y) = max(a, f_a(x) * f_a(y)) = max(a, a^{2-x-y})$, and $f_a(x \&_L y) = a^{1-max(0,x+y-1)}$. Both give a in the case when $x + y \leq 1$ and $a^{1-(x+y-1)}$ otherwise.

□

Now, for each formula φ with propositional variables in $\{p_1, ..., p_n\}$ we define a translation φ^* using one new propositional variable p_0. Let (for $i \in \{1, ..., n\}$):

$$\overline{0}^* \equiv p_0$$
$$p_i^* \equiv p_0 \vee p_i$$
$$(\varphi \& \psi)^* \equiv p_0 \vee (\varphi^* \& \psi^*)$$
$$(\varphi \rightarrow \psi)^* \equiv \varphi^* \rightarrow \psi^*$$

In particular $(\neg\varphi)^* \equiv \varphi^* \rightarrow p_0$. We will prove first a technical lemma, that will demonstrate the correspondence between e_L and e_P.

Lemma 2 Let e be an evaluation of propositional variables including p_0, with $e(p_0) > 0$. Let e', e'' be evaluations such that $e'(p_i) = max(a, e(p_i))$ and $e''(p_i) = f_a^{-1}(e'(p_i))$, where $a = e(p_0)$. Then for every formula φ not containing p_0 $f_a(e''_L(\varphi)) = e_P(\varphi^*)$.

Proof The proof goes by induction on the complexity of φ.

- Atomic formulas. For $\varphi \equiv 0$ it is obvious, if $\varphi \equiv p$ then
  $e_P(p^*) = e(p_0 \vee p) = max(a, e(p))$
  $f_a(e''_L(p)) = f_a(f_a^{-1}(max(a, e(p))) = max(a, e(p))$
- Composed formulas.
  - $\varphi \equiv \psi \& \chi$
    $e_P((\psi\&\chi)^*) = e_P(p_0 \vee (\psi^*\&\chi^*)) = max(a, e_P(\psi^*\&\chi^*)) = max(a, e_P(\psi^*) * e_P(\chi^*))$
    $f_a(e''_L(\psi\&\chi)) = f_a(max(0, e''_L(\psi) + e''_L(\chi) - 1))$, which by the induction assumption equals to $f_a(max(0, f_a^{-1}(e_P(\psi^*)) + f_a^{-1}(e_P(\chi^*)) + 1)) = max(a, a^{\log_a e_P(\psi^*)} * a^{\log_a e_P(\chi^*)})$
    $= max(a, e_P(\psi^*) * e_P(\psi^*))$
  - $\varphi \equiv \psi \rightarrow \chi$
    $e_P((\psi \rightarrow \chi)^*) = e_P(\psi^* \rightarrow \chi^*)$, which is 1 in the case when $e_P(\psi^*) \leq e_P(\chi^*)$ and $\frac{e_P(\chi^*)}{e_P(\psi^*)}$ otherwise
    $f_a(e''_L(\psi \rightarrow \chi)) = f_a(min(1, 1 - e''_L(\psi) + e''_L(\chi)))$, which is 1 in the case when $e''_L(\psi) \leq e''_L(\chi)$ and $f_a(1 - f_a^{-1}(e_P(\psi^*)) + f_a^{-1}(e_P(\chi^*))) = f_a(1 + \log_a e_P(\psi^*) - \log_a e_P(\chi^*)) = \frac{e_P(\chi^*)}{e_P(\psi^*)}$ otherwise.

□

Theorem 1 Let $\varphi^\dagger$ denote the formula $\neg\neg p_0 \rightarrow \varphi^*$. For each formula φ not containing p_0, φ is 1-tautology of Lukasiewicz's logic if and only if $\varphi^\dagger$ is a 1-tautology of product logic.

Proof If φ is 1-tautology of Łukasiewicz's logic then for every evaluation e, $e_L(\varphi) = 1$. This holds in particular for the derived evaluation e'' and thus $e_P(\varphi^*) = 1$ (the case when $e(p_0) = 0$ is obvious). Conversely if $\varphi^\dagger$ is 1-tautology of product logic, then for every evaluation e, $e_P(\neg\neg p_0 \rightarrow \varphi^*) = 1$, in particular for every e such that $e(p_0) > 0$, $e_P(\varphi^*) = 1$. We can see, that for every evaluation e, there is an evaluation $\overline{e}$ and a constant $0 < a < 1$ such that $e(p) = f_a^{-1}(max(a, \overline{e}(p)))$ for every propositional variable p. Let $\overline{e}(p_0) = a$, from $\overline{e}_P(\varphi^*) = 1$ follows $e_L(\varphi) = 1$ and therefore φ is 1-tautology of Lukasiewicz's calculus.

□

Altogether we proved that Łukasiewicz's logic has a faithful interpretation in product logic.

2.1 Recursive non-axiomatizability of product predicate logic

Now we investigate a language of predicate calculus consisting of predicates $P_1, P_2, ...$ and constants $c_1, c_2, ...$. Models are defined as usual (cf. [6, 2]), i.e. structures $\mathcal{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ where $M \neq \emptyset$, for each n-ary predicate symbol P, realization of P is a n-ary fuzzy relation r_P (function from M^n to [0,1]). And for each constant c, $m_c \in M$ is the realization of this constant.

In the following let $p_0 \equiv Q(c_0)$ be a new closed atomic formula. Now we present a similar translation * for the predicate calculus. It treats propositional connectives as above. For n-ary predicate symbol P define $(P(t_1,...,t_n))^* = p_0 \vee P(t_1,...,t_n)$, and finally put $((\forall x)\varphi)^* = (\forall x)\varphi^*$. We prove first a technical lemma similar to the propositional case. Observe the importance of the fact that our embedding f_a is a continuous mapping.

Lemma 3 Let φ be a closed formula not containing Q. Let $x \in [0,1]$. There is a model $\mathcal{M}$ of Łukasiewicz's predicate logic with $||\varphi||^{L}_{\mathcal{M}} = x$ if and only if there is a model $\mathcal{M}'$ of product predicate logic such that $||\varphi^*||^{P}_{\mathcal{M}'} = f_a(x)$.

Proof Let $\mathcal{M} = \langle M, (r_P)_P, (m_c)_c\rangle$ be a model of Łukasiewicz's predicate logic such that $||\varphi||^{L}_{\mathcal{M}} = x$. Pick $a > 0$ and construct a model $\mathcal{M}' = f_a(\mathcal{M}) = \langle M, (r'_P)_P, (m_c)_c\rangle$ by modifying $r'_P = f_a(r_P)$ and adding $r'_Q(c_0) = a$. Now it is easy to show by induction on complexity of φ that $||\varphi^*||^{P}_{\mathcal{M}'} = f_a(x)$. Let us show this just for the case when $\varphi \equiv (\forall x)\psi(x)$. Then $||\varphi||^{L}_{\mathcal{M}} = \inf_x\{||\psi||^{L}_{\mathcal{M}} = x\} = y$, $||\varphi^*||^{P}_{\mathcal{M}'} = \inf_x\{||\psi^*||^{P}_{\mathcal{M}'} = x\} = \inf_{f_a(x)}\{||\psi||^{L}_{\mathcal{M}} = x\} = f_a(\inf_x\{||\psi||^{L}_{\mathcal{M}} = x\} = f_a(y)$ ($f_a(x)$ is continuous). The other implication can be proven similarly.

□

Lemma 4 Let $\varphi^\dagger$ be again $\neg\neg p_0 \to \varphi^*$. Then φ is a 1-tautology of Łukasiewicz's predicate logic if and only if $\varphi^\dagger$ is a 1-tautology of the product predicate logic.

Proof Let φ be a 1-tautology of Łukasiewicz's predicate calculus. By the preceding lemma for every model $\mathcal{M}$, if $||\varphi^*||^{P}_{\mathcal{M}} = x$ then there is a model $\mathcal{M}'$ such that $||\varphi||^{L}_{\mathcal{M}'} = f_a^{-1}(x) = 1$. From this $x = 1$ and φ is 1-tautology of product predicate logic. Converse implication follows similarly from the above lemma.

□

As we know (see [10]), the set of 1-tautologies of Łukasiewicz's predicate calculus is Π_2-complete, hence not a Σ_2-set. From this fact and the above lemma we can conclude following theorem.

Theorem 2 The set of all 1-tautologies of the product predicate logic is not a Σ_2-set (and hence not recursively enumerable).

2.2 On monadic predicate logics

Ragaz [10] shows the following: (4.8.3) The set of all 1- tautologies of the monadic Łukasiewicz logic is a Π_1 set. (4.8.1) The set of all satisfiable closed formulas of this logic with at least four unary predicates is Π_1-complete. (A closed formula is satisfiable if it has the value 1 for at least one model.) In [11] he formulates the *problem* whether the set of all 1-tautologies of the logic in question is decidable, and, if not, if it is Π_1-complete. Thus we get immediately the following corollary for the monadic product logic:

Theorem 3 The set of satisfiable closed formulas of the monadic product logic with at least four predicates is not a Σ_1-set (i.e. it is not recursively axiomatizable).

Let us mention in passing some false statements in the literature. As Ragaz mentions in [11], Scarpellini's claim saying that Rutledge has shown decidability of the set of 1-tautologies of the monadic Łukasiewicz predicate logic is false (since Rutledge's system allows only quantification of a single variable). Gottwald claims ([2] p. 232) that Rutledge [12] has shown the axiomatizability of the set of 1-tautologies of the monadic Łukasiewicz predicate calculus, and (p. 237) that Ragaz has shown its undecidability. It follows from the quotations above that both claims are false.

3 Embedding Gödel's logic into extended product logic

Let us now enrich, as in [1], the language of the Gödel's logic by a new unary connective $\triangle$ such that $e(\triangle\varphi) = 1$ if $e(\varphi) = 1$ and $e(\triangle\varphi) = 0$ otherwise. By [1] the extension of the deductive system of the Gödel's logic by the following axioms:

(△**1**)	$\triangle\varphi \vee \neg\triangle\varphi$
(△**2**)	$\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$
(△**3**)	$\triangle\varphi \to \varphi$
(△**4**)	$\triangle\varphi \to \triangle\triangle\varphi$
(△**5**)	$\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$

and by the deduction rule

$$\frac{\varphi}{\triangle\varphi}$$

is a complete axiomatization of such extension of Gödel's logic. Similarly we can extend the deduction system of the product logic (see [5]) in the same way. In the following we will denote this extended deduction system by $P_\triangle$.

We introduce the notion of a $\triangle$-*product algebra* as a structure $\mathcal{A} = \langle A, \&_P, \to_P, \triangle, 0, 1\rangle$ such that $\langle A, \&_P, \to_P, 0, 1\rangle$ is a product algebra in the sense of [5] and it satisfies the following $\triangle$-axioms:

(△**i**)	$\triangle a \vee \neg\triangle a = 1$
(△**ii**)	$\triangle(a \to_P b) \leq (\triangle a \to_P \triangle b)$
(△**iii**)	$\triangle a \leq a$
(△**iv**)	$\triangle a \leq \triangle\triangle a$
(△**v**)	$\triangle(a \vee b) \leq (\triangle a \vee \triangle b)$
(△**vi**)	$\triangle 1 = 1$

First we will prove some basic facts, that will be used in the proof of completeness.

Lemma 5 a) $a \leq b$ implies $\triangle a \leq \triangle b$
b) $\triangle a = \triangle a \;\&_P\; \triangle a$
c) $\triangle(a \;\&_P\; b) = \triangle a \;\&_P\; \triangle b$

Proof a) If $a \leq b$ then $a \to_P b = 1$ holds in each product algebra, by $(\triangle vi)$ and $(\triangle ii)$ $1 = \triangle(a \to_P b) \leq \triangle a \to_P \triangle b$ and hence using the operation of taking the residuum ($u \leq (v \to_P w)$ iff $(u \&_P v) \leq w$) $\triangle a \leq \triangle b$.

b) $1 = a \to_P a \geq a$, from a) $\triangle(a \to_P a) \geq \triangle a$, again using $(\triangle ii)$ we obtain $\triangle a \to_P \triangle a \geq \triangle a$ which is the same as $\triangle a \geq \triangle a \;\&_P\; \triangle a$. Conversely $\triangle a \leq \triangle a \to_P \triangle a \;\&_P\; \triangle a$ and from the fact that in each product algebra $\neg\triangle a \leq \triangle a \to_P \triangle a \;\&_P\; \triangle a$ we can derive $1 = \triangle a \vee \neg\triangle a \leq \triangle a \to_P \triangle a \;\&_P\; \triangle a$ and hence $\triangle a \leq \triangle a \;\&_P\; \triangle a$.

c) $\triangle a, \triangle b \geq \triangle(a \;\&_P\; b)$ and from b) we get $\triangle(a \;\&_P\; b) = \triangle(a \;\&_P\; b) \;\&_P\; \triangle(a \;\&_P\; b)$, so together we can conclude $\triangle(a \;\&_P\; b) \leq \triangle a \;\&_P\; \triangle b$. The converse inequality $\triangle a \;\&_P\; \triangle b \leq \triangle(a \;\&_P\; b)$ is obvious.

$\square$

Definition 1 A *filter* over a $\triangle$-product algebra $\mathcal{A}$ is a subset $F \subseteq A$ which is a filter in the sense of product algebras (that means if $a \in F$ and $b \in F$ then also $a \;\&_P\; b \in F$ and if $a \in F$ and $a \leq b$ then also $b \in F$), satisfying the additional condition if $a \in F$ then also $\triangle a \in F$.

Lemma 6 The unit interval with truth functions $x \,\&_P\, y = x * y$, $x \rightarrow_P y = 1$ if $x \le y$ and $\frac{y}{x}$ otherwise, and the truth-function $\triangle$ such that $\triangle(1) = 1$, $\triangle(x) = 0$ for $x < 1$ is a linearly ordered $\triangle$- product algebra. The algebra of classes of provably equivalent formulas in $P_\triangle$ is a $\triangle$-product algebra.

Proof The proof is standard.

$\square$

Theorem 4 The deduction system $P_\triangle$ is a complete axiomatization of the set of 1-tautologies of logic $P_\triangle$.

To prove this completeness theorem we inspect the proof of the completeness theorem from [5] for the product logic in the following series of statements.

Sublemma 4.1 Let $\mathcal{A}$ be a $\triangle$-product algebra and let F be a filter. Define the corresponding equivalence $a \sim_F b$ iff $(a \rightarrow_P b) \in F$ and $(b \rightarrow_P a) \in F$. Then $\sim_F$ is a congruence, and the quotient algebra $\mathcal{A}/\sim_F$ is a $\triangle$-product algebra, which is linearly ordered if and only if F is an ultrafilter.

Proof We only have to verify that $a \sim_F b$ implies $\triangle a \sim_F \triangle b$. Assume $a \sim_F b$. Then $a \rightarrow_P b, b \rightarrow_P a \in F$, hence $\triangle(a \rightarrow_P b), \triangle(b \rightarrow_P a) \in F$, consequently $\triangle a \rightarrow_P \triangle b, \triangle b \rightarrow_P \triangle a \in F$ and we get $\triangle a \sim_F \triangle b$. The rest is as in [5].

$\square$

Sublemma 4.2 Let $\mathcal{A}$ be a $\triangle$-product algebra and $a \in A$, $a \neq 1$. Then there is an ultrafilter F on A not containing a.

Proof The construction is the same, only observe that if F is a filter not containing $c \rightarrow_P d$, then $F' = \{u | (\exists v \in F)(u \ge v \,\&_P\, \triangle(c \rightarrow_P d))\}$ is the smallest filter extending F and containing $c \rightarrow_P d$. In the following we will check that it's really the case (if $u_1, u_2, u \in F'$ then also $u_1 \,\&_P\, u_2, \triangle u \in F'$).

Assume $u_1, u_2 \in F'$. Then $u_i \ge v_i \,\&_P\, \triangle(c \rightarrow_P d)$ for suitable v_i and $i = 1, 2$, hence $u_1 \,\&_P\, u_2 \ge v_1 \,\&_P\, v_2 \,\&_P\, \triangle(c \rightarrow_P d) \,\&_P\, \triangle(c \rightarrow_P d) = v_1 \,\&_P\, v_2 \,\&_P\, \triangle(c \rightarrow_P d)$ by lemma 5 b). But $v_1 \,\&_P\, v_2 \in F$, and hence $u_1 \,\&_P\, u_2 \in F'$.

And if $u \in F'$ then $u \ge v \,\&_P\, \triangle(c \rightarrow_P d)$ for some $v \in F$. From this $\triangle u \ge \triangle(v \&_P \triangle(c \rightarrow_P d)) = \triangle v \,\&_P\, \triangle\triangle(c \rightarrow_P d) = \triangle v \,\&_P\, \triangle(c \rightarrow_P d)$ and hence $\triangle u \in F'$.

Now we start the construction of an ultrafilter not containing $1 \neq a \in A$. Start with $F = \{1\}$. Now if F is any filter not containing a and c, d are such that if neither $(c \rightarrow_P d)$ nor $(d \rightarrow_P c)$ belongs to F, create F', F'' as $F' = \{u | (\exists v \in F)(u \ge v \,\&_P\, \triangle(c \rightarrow_P d))\}$ and the same for F'' as the smallest filter extending F and containing $(d \rightarrow_P c)$. Let us suppose that a is member of both F' and F''. Then for some $v \in F$ $a \ge (v \,\&_P\, \triangle(c \rightarrow_P d)) \vee (v \,\&_P\, \triangle(d \rightarrow_P c)) \ge v \,\&_P\, (\triangle(c \rightarrow_P d) \vee \triangle(d \rightarrow_P c)) \ge v \,\&_P\, \triangle((c \rightarrow_P d) \vee (d \rightarrow_P c)) = v \,\&_P\, \triangle 1 = v \,\&_P\, 1 = v$, which means that $a \in F$, but this is a contradiction. Take for F that of F', F'' which doesn't contain a and iterate.

$\square$

Sublemma 4.3 Each $\triangle$-product algebra is a subdirect product of linearly ordered $\triangle$-product algebras.

The proof is the same as in [5].

Sublemma 4.4 If $\mathcal{A}$ is a $\triangle$-product algebra then $\triangle 1 = 1$. Moreover if $\mathcal{A}$ is linearly ordered, then $a \neq 1$ implies $\triangle a = 0$.

Proof If $a \neq 1$, then $\triangle a \neq 1$ (since $\triangle a \leq a$). Thus if $\mathcal{A}$ is linearly ordered from $(\triangle a \vee \neg\triangle a) = 1$ we get $\neg\triangle a = 1$ and hence $\triangle a = 0$.

□

Sublemma 4.5 If an identity $\tau = \sigma$, in the language of $\triangle$-product algebras, is valid in the unit interval algebra, then it is valid in all linearly ordered $\triangle$-product algebras.

Proof Use the proof from [5] observing that, by the preceding lemma, each linearly ordered product algebra extends uniquely a linearly ordered $\triangle$-product algebra; thus the representation using ordered Abelian groups works here too.

□

Now we can complete the proof of the completeness theorem as follows: if φ is a 1-tautology of $P_\triangle$ then the identity $\varphi = 1$ is valid in the unit interval $\triangle$-product algebra and hence in all linearly ordered $\triangle$-product algebras, thus, due to sublemma 4.3 in all $\triangle$-product algebras, among others in the algebra of classes of provably equivalent formulas of $P_\triangle$, which means that φ is provable in $P_\triangle$.

Theorem 5 Both Gödel's logic and its $\triangle$-extension are sublogics of $P_\triangle$.

Proof The theorem follows from the simple fact that the formula $(\varphi \to (\varphi \& \psi)) \vee \triangle(\varphi \to \psi)$ defines in $P_\triangle$ Gödel's implication, in other words $e_{P_\triangle}((\varphi \to (\varphi\&\psi)) \vee \triangle(\varphi \to \psi)) = e_G(\varphi \to \psi)$ for every evaluation e. This can be easily verified.

□

4 Computational complexity of propositional satisfiability

Let us define, analogously to the Boolean case, the set $\mathrm{SAT}_\circ$ for multi-valued logics Ł, P, G as the set of all formulas for which there is an evaluation of propositional variables e such that $e_\circ(\varphi) > 0$. It is well-known that the set of satisfiable formulas in Boolean logic SAT_{Boole} is NP-complete (see [3]). Mundici [9] proved that SAT_L is also NP-complete. In this section we shall establish the same for SAT_P.

First we will investigate the following translation of formulas:

Definition 2 Let φ be a formula with atoms $p_1, \ldots, p_n$ and let $I \subseteq \{p_1, \ldots, p_n\}$. Define a formula φ^I by induction on the logical complexity of φ as follows:

1. If $\varphi = p_i$ and $p_i \in I$ then put $\varphi^I := 0$. If $p_i \notin I$, put $\varphi^I := p_i$.
2. If φ is a constant then $\varphi^I := \varphi$.
3. Let $\varphi = \psi_1 \& \psi_2$.
   - If one of ψ_i^I is 0, put $\varphi^I := 0$.
   - If one of ψ_i^I is 1, put $\varphi^I := \psi_{3-i}^I$.
   - Otherwise put $\varphi^I := \psi_1^I \& \psi_2^I$.
4. Let $\varphi = \psi_1 \to \psi_2$.
   - If $\psi_1^I = 0$, put $\varphi^I := 1$.
   - If $\psi_1^I = 1$, put $\varphi^I := \psi_2^I$.
   - If $\psi_1^I \neq 0$ and $\psi_2^I = 0$, put $\varphi^I := 0$.
   - If $\psi_2^I = 1$, put $\varphi^I := 1$.
   - Otherwise put $\varphi^I := \psi_1^I \to \psi_2^I$.

Lemma 7 For any φ and I, φ^I is either a constant (0 or 1), or a formula not containing either 0 or 1.

Proof By inspection of Definition 2.

□

Now we present some statements that are valid for both product logic and Gödel's logic. In the following let $\bullet$ stand for P or G.

Lemma 8 Let φ be a formula not containing the constant 0. Then there is an evaluation e of propositional variables in φ such that $e_\bullet(\varphi) = 1$.

In particular, $\varphi \in \mathrm{SAT}_\bullet$.

Proof Assign to all atoms 1. Then the value of φ is 1 as well.

□

Theorem 6 *$SAT_\bullet \in NP$*

Proof

1. Given a formula φ that is composed from propositional variables $p_1, \ldots, p_n$, guess a subset $I \subseteq \{p_1, \ldots, p_n\}$.
2. Compute φ^I.
3. Accept if $\varphi^I \neq 0$.

If $\varphi \in \mathrm{SAT}_\bullet$ and $e_\bullet(\varphi) > 0$, then $I := \{p_i \mid e(p_i) = 0\}$ will not reduce φ to the constant 0 (as all rules in Definition 2 are consistent with the truth-tables of $\bullet$).

Conversely, if $\varphi^I \neq 0$ for some I then by lemmas 7, 8 $e_\bullet(\varphi) = 1$ for the assignment

$$e(p_i) = \begin{cases} 1 & \text{if } p_i \notin I \\ 0 & \text{if } p_i \in I \end{cases}$$

and hence $\varphi \in \mathrm{SAT}_\bullet$.

□

We have proved that both SAT_P and SAT_G are in NP (the result for Gödel's logic was already known (folklore)). Moreover, if we modify a given formula φ by adding double negation $(\neg\neg)$ before every propositional variable in φ we obtain a formula $\varphi^{\neg\neg}$ which is in $\mathrm{SAT}_\bullet$ if and only if formula φ is in SAT_{Boole} (as $\neg\neg a = 1$ for $a \neq 0$ in both P and G). Hence SAT_{Boole} reduces to $\mathrm{SAT}_\bullet$ and we have the following corollary.

Corollary 7 Both SAT_P and SAT_G are NP-complete.

We deduce few more corollaries.

Corollary 8 If $\varphi \in \mathrm{SAT}_P$ then, in fact, φ has the value 1 for some assignment.

Łukasiewisz's logic has a related property. Namely, the set of formulas for which there exists an assignment giving it the value 1 in L is in NP (by the proof of [9, Thm 2.4]).

Now let us denote by $\mathrm{TAUT}_\circ$ the set of all 1- tautologies of the corresponding logic.

Lemma 9 $\mathrm{TAUT}_P \in$ co-NP.

Proof We will see that the complement is in NP - let $\varphi \notin \mathrm{TAUT}_P$. This means by definition that there is an evaluation e of propositional variables such that $e_P(\varphi) < 1$. That exists if and only if there is an subset I of propositional variables in φ and a positive evaluation e' of φ^I ($e'(p_i) > 0$ for every p_i in φ^I) such that $e'_P(\varphi^I) < 1$. It is sufficient to put $e(p_i) = 0$ iff $p_i \in I$. Due to the isomorphism from section 2 $(\exists I)(\exists e \text{ positive})(e_P(\varphi^I) < 1)$ if and only if $(\exists I)(\exists e \text{ positive})(e_L(\varphi^I) < 1)$. Finally this holds if and only if $(\exists I)(\exists e \text{ positive})(e_L(\bigwedge_{p_i \in \varphi^I} p_i \wedge \neg(\varphi^I)) > 0)$.

Thus $\varphi \notin \mathrm{TAUT}_P$ iff $\bigwedge_{p_i \in \varphi^I} p_i \wedge \neg(\varphi^I) \in \mathrm{SAT}_L$ and hence $\mathrm{TAUT}_P \in$ co-NP.

□

The co-NP completeness of TAUT_P follows from the fact that $\varphi \in \mathrm{TAUT}_{Boole}$ if and only if $\varphi^{\neg\neg} \in \mathrm{TAUT}_P$.

Corollary 9 TAUT_P is an co-NP complete set.

All three sets $\mathrm{TAUT}_\circ$ are coNP-complete, and formal systems for any of the three logics is thus a non-deterministic acceptor of a coNP-complete set. The length-of-proofs question for any of L, G, P is therefore related to the NP $=_?$ coNP problem in the same way as it is for Boolean logic (see [4] for basic definitions and facts or [7] for a general background). Hence we propose to study the lengths of proofs in Ł, G and P with an ultimate goal to show that they cannot be polynomially bounded (in the lengths of formulas being proved) in any formal system. For Boolean case such lower bounds are known only for few formal systems, all weaker than the usual Hilbert-style system (cf. [7]).

The lower bounds in the Boolean case seem to be possible because the systems operate with formulas of various restricted types only (e.g. clauses in resolution). It would be interesting should it be possible to prove a lower bound for a natural formal system for one of Ł, G or P, that operates with all formulas.

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