A Mathematical Incompleteness in Peano Arithmetic *

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* *Editor's Note*: Since 1931, the year Godel's Incompleteness Theorems were published, mathematicians have been looking for a strictly mathematical example of an incompleteness in first-order Peano arithmetic, one which is mathematically simple and interesting and does not require the numerical coding of notions from logic. The first such examples were found early in 1977, when this *Handbook* was almost finished. We are pleased to be able to include a final chapter which presents the most striking example to date. It is a fitting conclusion since it brings together ideas from all Parts of the *Handbook*.

HANDBOOK OF MATHEMATICAL LOGIC
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1. An extension of the Finite Ramsey Theorem

In this chapter we present a recent discovery in mathematical logic. We investigate a reasonably natural theorem of finitary combinatorics, a simple extension of the Finite Ramsey Theorem. This chapter is mainly devoted to demonstrating that this theorem, while true, is not provable in Peano arithmetic.

The first examples of strictly mathematical statements about natural numbers which are true but not provable in PA (Peano arithmetic) were due to the first author (see PARIS [to appear]) and grew out of the work in PARIS and KIRBY [to appear]. The second author's contribution was to show that Paris's proof could be carried through with the particularly simple extension of the Finite Ramsey Theorem mentioned above (and stated precisely in 1.2).

Since we are going to work extensively with the partition calculus, the reader would be wise to consult pages 390–393 of Chapter B.3 for basic information and pages 393–395 for a proof of the Infinite Ramsey Theorem.

1.1. DEFINITION. We call a finite set H of natural numbers *relatively large* if $\mathrm{card}(H) \geq \min(H)$. Given natural numbers e, r, k and M, we use the notation

$$M \underset{*}{\longrightarrow} (k)_r^e$$

to mean that for every partition $P : [M]^e \to r$ there is a relatively large $H \subseteq M$ which is homogeneous for P and of cardinality at least k.

1.2. THEOREM. *For all natural numbers e, r and k there is an M such that $M \underset{*}{\longrightarrow} (k)_r^e$.*

Without the $*$ under the arrow which makes the homogeneous sets relatively large, this would just be the Finite Ramsey Theorem. The Finite Ramsey Theorem is provable in Peano Arithmetic. Our proof of 1.2 will use the Infinite Ramsey Theorem, and cannot be carried out in PA.

1.3. MAIN THEOREM. *The combinatorial principle of* 1.2 *is not provable in Peano Arithmetic.*

For the reader who is not used to working in PA, and so does not even see how to formulate 1.2 in PA, we would remark that PA is equivalent (for statements about natural numbers) to the result of replacing the axiom of infinity by its negation in the usual axioms ZF of set theory (see Chapter B.1), and 1.2 can be formulated in this theory directly, without any coding.

2. Proofs of 1.2 and 1.3

We first prove 1.2. Fix e, r and k, and suppose there were no M of the desired kind. Call P a counterexample for M if P is a partition of $[M]^e$ into r pieces with no relatively large homogeneous set of size at least k. We may view the set of counterexamples as a finitely branching infinite tree. That is, if P and P' are counterexamples for M and M' respectively, we put P below P' in our tree just in case $M < M'$ and P is the restriction of P' to $[M]^e$. By König's Lemma there is a $P : [\omega]^e \to r$ such that for every M, the restriction of P to $[M]^e$ is a counterexample for M. By the Infinite Ramsey Theorem, there is an infinite $H \subseteq \omega$ homogeneous for P. But then by choosing M large enough (compared to k and $\min(H)$) we see that $H \cap M$ is, after all, a relatively large homogeneous set for $P \restriction [M]^e$ of size at least k. $\square$

Looking ahead to Section 3, we point out that, for each e, the above proof can be formalized in PA. (The proof on pages 393–395 of $\omega \to (\omega)^e_r$ is naturally formalized, by induction on e, in restricted-($\Pi^0_\infty -$ CA), which is conservative over PA (see page 940).) Thus, for every e,

$$\mathrm{PA} \vdash \forall r, k\, \exists M\, (M \underset{*}{\longrightarrow} (k)^e_r).$$

We now begin the proof of 1.3, which will take up the remainder of this section. We define a certain theory T in 2.1 and then show $\mathrm{Con}(T) \to \mathrm{Con}(\mathrm{PA})$ is provable in PA. The proof will be concluded by showing, in PA, that the combinatorial principle of 1.2 implies $\mathrm{Con}(T)$.

For the purpose of the following we identify finite subsets of ω with finite increasing sequences from ω. The theory T is expressed in the language of PA plus infinitely many new constant symbols $c_0, c_1, \ldots$.

2.1. Definition. The axioms of T are as follows:

(i) The usual recursive defining equations for $+, \times, <$, plus the induction axioms but only for limited formulas.

(ii) For each $i = 0, 1, \ldots$, the axiom $(c_i)^2 < c_{i+1}$.

(iii) For each finite subset $\boldsymbol{i} = i_1, \ldots, i_r$ of ω, let $c(\boldsymbol{i}) = c_{i_1}, \ldots, c_{i_r}$.

For each $i < \boldsymbol{k}, \boldsymbol{k}'$ and each limited formula $\psi(\boldsymbol{y};\boldsymbol{z})$ (where $\boldsymbol{k}, \boldsymbol{k}'$ and $\boldsymbol{z}$ all have the same length) we have the axiom:

$$\forall \boldsymbol{y} < c_i\, [\psi(\boldsymbol{y}; c(\boldsymbol{k})) \leftrightarrow \psi(\boldsymbol{y}; c(\boldsymbol{k}'))].$$

2.2. PROPOSITION. Con(T) *implies* Con(PA).

PROOF. Let $\mathfrak{A} \models T$ and let I be the initial segment of $\mathfrak{A}$ of those $a < c_i$ for some $i \in \omega$. By (ii), I is closed under $+$ and $\times$. It will be enough to show the following.

2.3. CLAIM. $\mathfrak{J} = \langle I, +, \times, < \rangle$ is a model of PA.

For each formula $\theta(\boldsymbol{y})$ from the language of PA, define a limited formula $\theta^*(\boldsymbol{y};\boldsymbol{z})$ as follows. Write θ in prenex normal form, say $\exists x_1 \cdots \forall x_r\, \varphi(\boldsymbol{x};\boldsymbol{y})$ where φ is quantifier free. Then $\theta^*(\boldsymbol{y}; z_1, \ldots, z_r)$ is $\exists x_1 < z_1 \cdots \forall x_r < z_r\, \varphi(\boldsymbol{x};\boldsymbol{y})$.

2.4. CLAIM. Given $i < \boldsymbol{k}$, $\boldsymbol{a} < c_i$, and $\theta(\boldsymbol{y})$, where $\boldsymbol{k}, \boldsymbol{a}$ and $\boldsymbol{y}$ are all of the appropriate length,

$$\mathfrak{J} \models \theta(\boldsymbol{a}) \quad \text{if and only if} \quad \mathfrak{A} \models \theta^*(\boldsymbol{a}; c(\boldsymbol{k})).$$

Notice that 2.3 is an immediate consequence of 2.4 since part (i) of 2.1 guarantees that for all θ, $\mathfrak{A}$ will satisfy induction for θ^*. Then proof of 2.4 proceeds by induction on θ. Suppose $\theta(\boldsymbol{y})$ is $\exists x\, \psi(x, \boldsymbol{y})$. Thus $\theta^*(\boldsymbol{y};\boldsymbol{z})$ is $\exists x < z_1\, \psi^*(x, \boldsymbol{y}; z_2, \ldots, z_r)$. So $\mathfrak{J} \models \theta(\boldsymbol{a})$ iff for some b in I and some $\boldsymbol{j}$ (where $\min(\boldsymbol{j})$ is large) $\mathfrak{A} \models \psi^*(b, \boldsymbol{a}; c(\boldsymbol{j}))$, which happens iff for some $\boldsymbol{k}'$ (again, where $\min(\boldsymbol{k}')$ is large) $\mathfrak{A} \models \theta^*(\boldsymbol{a}; c(\boldsymbol{k}'))$ which, by 2.1(iii), is the case iff $\mathfrak{A} \models \theta^*(\boldsymbol{a}; c(\boldsymbol{k}))$. □

The attentive reader should observe that the proof of 2.2 can be formalized in PA (in a way similar to Section 6 of Chapter D.1). Also, one should notice that for the purposes of the above proof, we can weaken 2.1(iii) to those limited formulas $\psi(\boldsymbol{y};\boldsymbol{z})$ of the form $\theta^*(\boldsymbol{y};\boldsymbol{z})$ for some $\theta(\boldsymbol{y})$

2.5. PROPOSITION. *The combinatorial principal of* 1.2 *implies* Con(T).

By Gödel's Second Incompleteness Theorem (see page 825), 2.5 and 2.2 yield our main theorem, provided of course that 2.5, like 2.2, is proved in PA.

Before beginning the proof of 2.5, we point out a few facts about partitions.

2.6. LEMMA. *Given partitions P_0 and P_1 of $[M]^e$ into r_0 and r_1 pieces, there is a partition P of $[M]^e$ into $r_0 \cdot r_1$ pieces such that for $H \subseteq M$, H is homogeneous for P iff H is homogeneous for both P_0 and P_1.*

PROOF. Let $P(\boldsymbol{a}) = \langle P_0(\boldsymbol{a}), P_1(\boldsymbol{a})\rangle$. □

2.7. LEMMA. *A set $H \subseteq M$ is homogeneous for a partition P of $[M]^e$ iff every subset of H of size $e+1$ is homogeneous for P.*

PROOF. Let $\boldsymbol{a} = a_1, \dots, a_e$ be the first e elements of H. Pick $\boldsymbol{b} = b_1, \dots, b_e$ so that $P(\boldsymbol{a}) \neq P(\boldsymbol{b})$ and so that $b_1 + \cdots + b_e$ is minimized. If i is the least index such that $a_i \neq b_i$, then $\{a_1, \dots, a_i, b_i, \dots, b_e\}$ is not homogeneous and of size $e+1$. □

We define $\sqrt{r}$ to be the first natural number s such that $s^2 \geq r$. Notice that for most r (i.e., for $r \geq 7$), $r \geq 1 + 2\sqrt{r}$.

2.8. LEMMA. *Given $P : [M]^e \to r$ there is a $P' : [M]^{e+1} \to (1 + 2\sqrt{r})$ such that for all $H \subseteq M$ of cardinaltiy $> e+1$, H is homogeneous for P iff H is homogeneous for P'.*

PROOF. Let $s = \sqrt{r}$. Define functions Q (for quotient) and R (for remainder) both mapping $[M]^e$ into s by the equation $P(\boldsymbol{a}) = s \cdot Q(\boldsymbol{a}) + R(\boldsymbol{a})$. For $\boldsymbol{b} = b_1, \dots, b_e, b_{e+1}$ in $[M]^{e+1}$, let $\boldsymbol{b}' = b_1, \dots, b_e$. We now define our desired P' on $[M]^{e+1}$ by:

$$P'(\boldsymbol{b}) = \begin{cases} 0 & \text{if } \boldsymbol{b} \text{ is homogeneous for } P, \\ \langle 0, R(\boldsymbol{b}')\rangle & \text{if } \boldsymbol{b} \text{ is homogeneous for } Q \text{ but not for } P, \\ \langle 1, Q(\boldsymbol{b}')\rangle & \text{otherwise,} \end{cases}$$

Let H be homogeneous for P' of cardinality $> e+1$, and let $\boldsymbol{c}$ be the first $e+1$ members of H. We need to see that $P'(\boldsymbol{c}) = 0$ to verify that H is homogeneous for P, by 2.7. Note that for each $\boldsymbol{a}$ in $[\boldsymbol{c}]^e$ there is a $\boldsymbol{b}$ in $[H]^{e+1}$ such that $\boldsymbol{b}' = \boldsymbol{a}$. Suppose $P'(\boldsymbol{c}) = \langle 1, i\rangle$. Then, by the previous remark, $Q(\boldsymbol{a}) = i$ for all $\boldsymbol{a}$ in $[\boldsymbol{c}]^e$ so that $\boldsymbol{c}$ is homogeneous for Q, contradicting the definition of P'. So suppose $P'(\boldsymbol{c}) = \langle 0, j\rangle$ so that $\boldsymbol{c}$ is Q homogeneous, say $Q(\boldsymbol{a}) = i$ for all $\boldsymbol{a}$ in $[\boldsymbol{c}]^e$. But then $P(\boldsymbol{a}) = s \cdot i + j$ for all such $\boldsymbol{a}$ so that $\boldsymbol{c}$ is homogeneous for P, again contradicting the definition of P'. □

2.9. LEMMA. *Suppose we are given n partitions $P_i : [M]^{e_i} \to r_i$, all $i < n$. Let $e = \max_i e_i$ and $r = \Pi_i \max(r_i, 7)$. There is a partition $P : [M]^e \to r$ such that for all $H \subseteq M$ of cardinality $> e$, H is homogeneous for P iff H is homogeneous for all the P_i.*

PROOF. Combine 2.6, 2.8 and the remark preceeding 2.8. □

We now state a combinatorial principle which is tailored to imply Con(T). Parts (ii) and (iii) of 2.10 correspond to the similar parts of 2.1. There is no 2.10(i). After showing that 2.10 implies Con(T), we will return to derive 2.10 from 1.2.

2.10. PROPOSITION. *For all e, k, r there is an M such that for any family $\langle P_\xi ; \xi < 2^M \rangle$ of partitions $P_\xi : [M]^e \to r$, there is an X of cardinality $\geq k$ such that:*

(ii) *if $a, b \in X$ and $a < b$, then $a^2 < b$,*
(iii) *if $a \in X$ and $\xi < 2^a$, then $X \sim (a+1)$ is homogeneous for P_ξ.*

2.11. CLAIM. 2.10 implies Con(T).

PROOF. Given a finite subset S of T, let $c_0, \ldots, c_{k-1}$ be all constants appearing in S. We will use 2.10 to show that S has a model of the form $\langle \omega ; +, \times, <, x_0, \ldots, x_{k-1} \rangle$, where $x_0, \ldots, x_{k-1}$ are the first k elements of the set X given by 2.10. This model clearly satisfies (i) of 2.1 so we need only worry about those axioms of the forms (ii) and (iii) in S. Part (ii) of 2.10 takes care of the axioms of form (ii) automatically, so we only need to set up our partitions to handle those of form (iii).

We may view each $\xi \in \omega$ as coding a finite increasing sequence $\boldsymbol{a}(\xi)$ from ω in such a way that all sequences from b are coded by some $\xi < 2^b$. Given a limited formula $\psi(\boldsymbol{y}; \boldsymbol{z})$ and a sequence $\boldsymbol{a}(\xi)$ of the same length as $\boldsymbol{y}$, we obtain a partition $F_{\psi,\xi} : [M]^{e'} \to 2$, where e' is the length of $\boldsymbol{z}$ defined by $F_{\psi,\xi}(\boldsymbol{c}) = 0$ if $\psi(\boldsymbol{a}(\xi); \boldsymbol{c})$, and $= 1$ otherwise.

Consider, for the moment, fixed M and ξ. For each axiom of type (iii) occurring in S there is a corresponding limited formula $\psi(\boldsymbol{y}; \boldsymbol{z})$ and hence a corresponding partition $F_{\psi,\xi}$. By 2.9 we may combine these into a single partition $P_\xi : [M]^e \to r$, where e and r depend only on S, not on ξ or M. Now using 2.10, choose M so large that (ii) and (iii) hold for some $X \subseteq M$

for the family $\langle P_\xi ; \xi < 2^M \rangle$, and $\mathrm{card}(X) \geq k + e$. Now choosing $x_0, \ldots, x_{k-1}$ as described above, we see that all axioms of type (iii) are satisfied. □

Our attentive reader will have noticed that since $\langle \omega; +, \times, < \rangle$ has a primitive recursive satisfaction relation for limited formulas, we can prove in PA (or even PRA) that this structure satisfies (i) of 2.1. Hence the above proof can be carried out in PA.

All that remains is for us to prove (in PA) that 1.2 does imply 2.10. To do this we need a method for obtaining homogeneous sets which grow fast. We are indebted to F. Abramson for some of the following arguments which have simplified our original proof.

For any function g, let $g^{(x)}$ be g composed with itself x times. Let $f_0(x) = x + 2$ and let $f_{n+1}(x) = (f_n)^{(x)}(2)$. The reader can check that $f_1(x) \geq 2x$, $f_2(x) \geq 2^x$, $f_3(x) \geq \beth_x$, where $\beth_x = 2^{2^{\cdot^{\cdot^{2}}}}$, a stack of x 2's, and so on for $f_4, f_5, \ldots$. Readers familiar with the Ackermann function will realize that each f_n is primitive recursive and that every primitive recursive function is eventually dominated by some f_n, but these facts will not be used below.

2.12. LEMMA. *For every p there is a $Q : [M]^1 \rightarrow p + 1$ such that if X is homogeneous for Q and of cardinality at least 2, then* $\min(X) \geq p$.

PROOF. Let $Q(a) = \min(a, p)$. □

We now come to two lemmas which use relatively large homogeneous sets.

2.13. LEMMA. *For each m there is a partition $R : [M]^2 \rightarrow r$ (where r depends only on m) such that if $X \subseteq M$ is relatively large and homogeneous for R and of cardinality > 2, then for every $x, y \in X$, $x < y$ implies $f_m(x) < y$.*

PROOF. For each $i \leq m$, let $P_i(a, b) = 0$ if $f_i(a) < b$; $= 1$ otherwise Let $p = f_m(3)$ and let Q be as in 2.12 for this p. Use 2.9 to combine all of these into $R : [M]^2 \rightarrow r$. Let X be relatively large and homogeneous for R. Let $a = \min(X)$, $b = \max(X)$. By induction on $i \leq m$, it is easy to show first that $f_i(a) < b$ (this is where you use $\mathrm{card}(X) \geq a$) and second that $f_i(x) < y$ for all x, y in X, $x < y$, by homogeneity. □

2.14. LEMMA. *Let $P : [M]^e \rightarrow s$ $(e \geq 2)$ and m be given. There is a $P^* : [M]^e \rightarrow s'$, where s' depends only on m, e and s, such that if there is a relatively large $Y \subseteq M$ homogeneous for P^* of cardinality $> e$, then there is* *an $X \subseteq M$ such that X is homogeneous for P and* card(X) *is at least* $e+1$ *and* $f_m(\min(X))$.

PROOF. Let $h(a)$ be the largest x such that $f_m(x) \leq a$. For $\boldsymbol{a} = a_1, \ldots, a_e$, let $\boldsymbol{h}_a = h(a_1), \ldots, h(a_e)$. Let $S(\boldsymbol{a}) = P(\boldsymbol{h}_a)$ if $\boldsymbol{h}_a$ is an e-tuple (i.e., if $h(a_1) < h(a_2) < \cdots < h(a_e)$); $S(\boldsymbol{a}) = s$ otherwise. Thus $S : [M]^e \to s+1$. Let R be as in 2.13. Use 2.9 to combine R, S into $P^* : [M]^e \to s'$. Let Y be given as in the statement of our lemma, and let X be the image of Y under h. The partition R promises us that h is one-one on Y so that card$(X) =$ card$(Y) \geq \min(Y)$. But the definition of h implies that $f_m(\min(X)) \leq \min(Y)$ so card$(X) \geq f_m(\min(X))$ as desired. □

2.15. PROPOSITION. *The combinatorial principle of* 1.2 *implies that of* 2.10.

PROOF. We are given e, k and r, and must produce an M as in 2.10. Find a p so that for all $a \geq p$, $f_3(a)$ is reasonably big as compared with e, r, k and a. We will make this precise in the last paragraph of the proof; for now, just note that $f_3(y) \geq \beth_y$. Let $e' = 2e+1$.

Now given any M and any family $P_\xi : [M]^e \to r$ for $\xi < 2^M$, define a new $S : [M]^{e'} \to 2$ by: $S(a, \boldsymbol{b}, \boldsymbol{c}) = 0$ if $P_\xi(\boldsymbol{b}) = P_\xi(\boldsymbol{c})$ for all $\xi < 2^a$; $S(a, \boldsymbol{b}, \boldsymbol{c}) = 1$ otherwise. Let Q be as in 2.12 and R as in 2.13 for $m = 2$. Use 2.9 to combine Q, R and S into a single P and then use 2.14 to obtain $P^* : [M]^{e'} \to s'$. The number s' depends only on $e' = 2e+1$ and on p. We now apply the combinatorial principle 1.2. Find an M such that $M \xrightarrow[*]{} (e'+1)^{e'}_{s'}$. By 2.12 there is an $X \subseteq M$ which is homogeneous for Q, R and S with card$(X) \geq f_3(\min(X))$. Since X is homogeneous for Q, $\min(X) \geq p$. Since X is homogeneous for R, and since $f_2(y) \geq y^2$ for those y big enough to be in X, X satisfies 2.10(ii).

To verify 2.10(iii), we replace X by $X' = X \sim \boldsymbol{d}$, where $\boldsymbol{d} = d_1, \ldots, d_e$ are the last e elements of X. Let $i_\xi = P_\xi(\boldsymbol{d})$. If we show that for all $a < b_1 < \cdots < b_e$ in X' and all $\xi < 2^a$, $P_\xi(\boldsymbol{b}) = i_\xi$, we will have shown that X' satisfies 2.10. To show this it suffices to show that $S(a, \boldsymbol{b}, \boldsymbol{c}) = 0$ for some (hence, by homogeneity, for every) $1+2e$ tuple $a, \boldsymbol{b}, \boldsymbol{c}$ from X. Let $a = \min(X)$ and consider consecutive e-tuples from $X \sim (a+1)$. Our choice of p earlier should be such that there are more than $r^{(2^a)}$ such e-tuples for then we can find e-tuples $\boldsymbol{b}, \boldsymbol{c}$ such that $P_\xi(\boldsymbol{b}) = P_\xi(\boldsymbol{c})$ for all $\xi < 2^a$, as desired. □

3. Refinements

In the proof of our main theorem we relied on various proof-theoretic results, in particular, on Gödel's Second Incompleteness Theorem. It is possible, however, to prove our main theorem using only model-theoretic methods. This is the approach taken in PARIS [to appear], where a general model-theoretic methodology (called indicator functions) for producing such results is developed.

On the other hand, 1.2 is actually equivalent, in PA, to a well-known proof-theoretic principle, and our proof has the advantage of making this fairly obvious. Recall, from page 849, the definition of RFN_{Σ_1}, the statement of number theory expressing the statment "For all Σ_1 sentences ψ, if $\mathrm{PA} \vdash \psi$, then ψ".

3.1. THEOREM. *It is a theorem of* PA *that* RFN_{Σ_1} *is equivalent to the combinatorial principle of* 1.2.

PROOF. After the proof of 1.2 we mentioned that

$$\text{for all } e, r, k, \quad \mathrm{PA} \vdash \exists M\, (M \underset{*}{\longrightarrow} (k)^e_r).$$

This fact, which we indicated how one would verify, is itself a theorem of PA. An application of RFN_{Σ_1} gives 1.2.

Assume 1.2 and let us prove RFN_{Σ_1}. Let ψ be a Σ_1 sentence. We prove that if $\neg\psi$, then $\mathrm{Con}(\mathrm{PA} + \neg\psi)$. The proof of 2.5 shows that if ψ is false in ω, then $\mathrm{Con}(T + \neg\psi)$, using 1.2. But the proof of 2.2 shows that $\mathrm{Con}(T + \neg\psi)$ implies $\mathrm{Con}(\mathrm{PA} + \neg\psi)$. □

For our final result, define a recursive function f by

$$f(e) = \text{the least } M \text{ such that } M \underset{*}{\longrightarrow} (e+1)^e_e.$$

3.2. THEOREM. *If g is a (description of a) recursive function, and if* $\mathrm{PA} \vdash$ "*g is total*", *then for all sufficiently large e, $f(e) > g(e)$.*

PROOF. Let S be a finite subset of T and let $c_0, \ldots, c_{k-1}$ be the constants appearing in S. As the proof of 2.5 (in particular that of 2.11) shows, we may interpret these constants so as to make ω a model of S. By examining that proof, one can see that for all large enough e, we can in fact interpret $c_0, \ldots, c_{k-1}$ using members of the interval $(e, f(e))$. If $g(e) \geq f(e)$ for infinitely many e, the above would show the consistency of T plus the following axioms in a new constant e:

$$e < c_0; \qquad \neg \exists x \leq c_i \, (g(e) \simeq x) \quad \text{for all } i \leq \omega.$$

By the proof of 2.2 we obtain the consistency of PA + $\exists e \, (g(e)$ is not defined). □

We wish to thank the editor for almost forcing us to write this chapter, for typing it himself, and for a number of minor changes, provided he accepts responsibility for all misprints.

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