Building models by games pt. 3

Ommiting types

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- If those properties are not contradictory, can we always add such a tuple into a structure while preserving valid formulas?
- Under some formalization of these notions the answer is **yes**!

Types

Definition (n-type)

*Let T be an L-theory, $n \geq 1$ and $x_1, \ldots, x_n$ are variables, then an n-**type** is a set $\Phi(x_1, \ldots, x_n)$ of L-formulas with free variables $x_1, \ldots, x_n$, such that*

$$\forall \Phi_0 \subseteq_{fin} \Phi : T \models (\exists \overline{x}) \bigwedge_{\phi \in \Phi} \phi(\overline{x}).$$

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- In words 'an n-type is a set of properties of a tuple that is always finitely satisfied'.

Realizing types

- We say that an n-type $\Phi(\overline{x})$ is **realized** if there is a structure $\mathcal{A} \models T$ and $\overline{a} \in A^n$ such that for all $\phi(x) \in \Phi(x)$

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- This is the most natural setting for studying types.

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Theorem (Realizing types)

Let $\mathcal{A}$ be an L-structure. Let $\Gamma = \{\Phi_0, \Phi_1, \dots\}$ be a set of types in the theory $T = Th_A(\mathcal{A})$, then there is some $\mathcal{B} \models T$ which realizes every type in Γ. $\mathcal{B}$ is an elementary extension of $\mathcal{A}$.

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Proof

Let

$$T' = T \cup \bigcup_{\Phi \in \Gamma} \Phi(c^{\Phi}),$$

where c^{Φ} are new constants. This theory is consistent by compactness. □

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- Let $T = \mathrm{Th}_{\mathbb{R}}((\mathbb{R}, +, \cdot, -, 1, 0, <))$, then the type $\Phi(x) = \{(0 < x < 1/2), (0 < x < 1/3), (0 < x < 1/4), \dots\}$ is realized in any non-archimedean extension of $\mathbb{R}$, for example
  $$\mathbb{R}(\omega) \models \Phi(\omega),$$
  where ω is some infinitesimal element.
- This proves 'being an archimedean field' is not a first-order property.

Isolated types

- We say that a type $\Phi(\bar{x})$ in an L-theory T is **isolated** or **principal** if there is a L-formula $\phi(\bar{x})$ such that

$$T \models (\forall \bar{x}) \left(\phi(\bar{x}) \to \bigwedge_{\psi \in \Phi} \psi(\bar{x}) \right).$$

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  - For $T = \mathrm{Th}_{\mathbb{Z}}((\mathbb{Z}, +, \cdot, -, 1, 0))$ the type $\Phi(x) = \{(x \cdot 2 = 2), (x \cdot 3 = 3), (x \cdot 4 = 4), \ldots\}$, is isolated by the formula $\phi(x) = (x = 1)$.
- Notice that for every $\mathcal{A} \models T$ and every type $\Phi(\bar{x})$ in T isolated by one of its elements we have that Φ is already realized in $\mathcal{A}$.

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- **!** Note that we can't expect to omit general isolated types.
- Finding models of T which omit some types can lead to interesting results.
- We would want some general theorem that would let us prove the existence of some model of T omitting specific types.

The omitting types theorem

Theorem (Omitting types)

Let L be a countable language. Let T be an L-theory. Let $\Gamma = \{\Phi_i; i < \omega\}$ be a set of non-isolated types. Then T has a model which omits every type in Γ.

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Lemma (Lemma on constants)

Let T be an L-theory and $\phi(\overline{x})$ an L-formula. Let $\overline{c}$ be a tuple of distinct constants not in L. Then

$$T \vdash \phi(\overline{c}) \Leftrightarrow T \vdash (\forall \overline{x})\phi(\overline{x}).$$

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Proof

Model theoretically: The constants can be interpreted in any way in each model of T. This means the formula $\phi(\overline{x})$ holds for every tuple of elements in every models of T. Therefore its universal closure is also true in every model of T.

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Proof theoretically: By induction on the complexity of the proof. □

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$$p \in N$$
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$$|p| < \aleph_0, \text{ and } T \cup p \text{ has a model.}$$

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We will show that omitting every type in Γ is N-enforceable.

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Making $\bigwedge_{i<\omega} \Phi_i(\overline{c})$ fail is enough to omit all the types, since we have proved that it is enforceable that every element in the compiled structure is named by countably many constants.

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All the $\exists$-player needs to do is to play a condition $p \cup \{\neg\phi(\bar{c})\}$ for some $\phi \in \Phi_i$ for every $i < \omega$.

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From the construction of N this means that $T \cup p \vdash \phi(\overline{c})$ for each $\phi \in \Phi_i$. Now the lemma on constants implies that $T \cup p \vdash (\forall x)\phi(\overline{x})$, which implies $T \vdash (\bigwedge_{\psi \in p} \psi) \to (\forall \overline{x})\phi(\overline{x})$, and since each $\psi \in p$ contains no free variables we have that $T \vdash (\forall \overline{x}) \left(\left(\bigwedge_{\psi \in p} \psi \right) \to \phi(x) \right)$. Therefore Φ_i is isolated. A contradiction. □