The midsequent theorem and witnessing

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  - an upper part which uses only structural and propositional inferences
  - a sequent S' which is the lower sequent of the last propositional inference
  - a lower part which uses only structural and quantifier inferences
- This can be then used to provide some witnessing theorems which are frequently used in the context of bounded arithmetic.

The statement

Theorem (The midsequent theorem)

Let S be a sequent consisting of formulas in prenex form which is provable in LK. Then there is cut free LK-proof P of S which contains a sequent S' (called the midsequent) satisfying:

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- *Every inference above S' is either structural or propositional inference*
- *Every inference below S' is either structural or quantifier inference*

The proof 1/4

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We already know, that there exists a cut free proof P of S, we can also assume that only sequents of the form $A \to A$ were used as initial sequents, where A is atomic.

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Let I be an inference instance in P, we define

$$\mathrm{ord}_P(I) = \text{number of propositional inferences below } I$$

and

$$\mathrm{ord}(P) = \sum_{I \text{ in } P} \mathrm{ord}_P(I).$$

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We proceed in constructing the LK-proof from the statement by induction on $\mathrm{ord}(P)$.

The proof 2/4

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Case $\text{ord}(P) = 0$: While in this case there is no propositional inference found below any quantifier instance, the sequenct S_0—defined as the lower sequent of the lowest propositional inference—might still contain formulas with quantifiers.

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From the assumption on the proof P, the quantifier formula(s) could have only been introduced using weakenings. But since the end-sequent S is prenex and the proof is cut free, there were no propositional inferences applied to any of them. So the weakening can be "postponed" after S_0 which finished this case.

The proof 3/4

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Case $\text{ord}(P) > 0$: Now there exists some quantifier inference I under which the uppermost logical inference is a propositional inference I'.We will lower the order of P by exchanging the positions of I and I'. We restrict ourself here to the case where I is $\forall$: right so we have

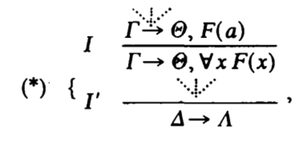
$$(*)\ \left\{\begin{array}{ll} I & \dfrac{\Gamma \to \Theta, F(a)}{\Gamma \to \Theta, \forall x F(x)} \\ & \quad\vdots \\ I' & \dfrac{\qquad\qquad}{\Delta \to \Lambda} \end{array}\right. ,$$

where $(*)$ contains only structural inferences.

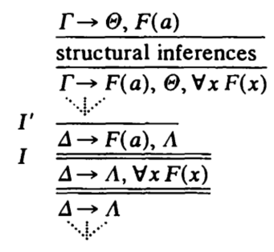
The proof 4/4

Proof cont.

The rearrangement in such a case looks like this:



$\rightarrow$



□

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Let T be a universal theory in the language L, $\varphi(x, y)$ a quantifier free L-formula and let

$$T \vdash (\forall x)(\exists y)\varphi(x, y),$$

then there exist L-terms $t_1, \dots, t_n$ such that

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- Remark: If L contains no terms or constants, the situation becomes trivial, because the terms are therefore simply variables, and therefore for any universal L-theory T we have that $T \vdash (\forall x)(\exists y)\varphi(x, y)$ implies $T \vdash (\forall x)\varphi(x, x)$. (e.g. the theory of graphs)

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Proof.

If $T \vdash (\forall x)(\exists y)\varphi(x, y)$ then there is some finite subset $\Gamma \subseteq T$ such that the sequent $\Gamma \to (\forall x)(\exists y)\varphi(x, y)$ is valid a therefore there is an LK-proof of it, called P, with a midsequent S'.

Herbrand's theorem – the proof 2/2

Proof cont.

Since S' is in P transformed into $\Gamma \to (\forall x)(\exists y)\varphi(x, y)$ by structural and quantifier inferences it has to be of the form:

$$S' : \quad \gamma_0(\bar{a}), \ldots, \gamma_n(\bar{a}) \to \varphi(b_1, t_1), \ldots, \varphi(b_n, t_n).$$

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$$S'' : \quad \Gamma \to \varphi(b_1, t_1), \ldots, \varphi(b_n, t_n),$$

and by weakening

$$S''' : \quad \Gamma, b_1 = b_2, b_1 = b_3, \ldots, b_1 = b_n \to \varphi(b_1, t_1), \ldots, \varphi(b_n, t_n),$$

from which the sequent $\Gamma \to \varphi(b, t_1(b)), \ldots, \varphi(b, t_n(b))$ logically follows using the equality axioms. □

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Let $T = \mathrm{RCF}$, the theory of real closed fields. One of the axioms of RCF is the existence of a cube root. So we trivially have

$$T \vdash (\forall x)(\exists y)(y^3 = x).$$

However, the language of RCF is the language of rings, so the only terms in L_{RCF} are polynomials with integer coefficients, which for cannot serve as an witness for y when $x := 2 \in \mathbb{R} \models \mathrm{RCF}$ and so the Herbrand disjunction cannot be provable in RCF.

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- Can be circumvented by adding a function symbol $\text{cbroot}(-)$ and the axiom $(\forall x)\text{cbroot}(x)^3 = x$.

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- Let $\varphi(x, p)$ be a system of polynomial equations with parameter p written out as a formula.
- We can see that if the theory

$$T \vdash (\forall p)(\exists x)\varphi(x, p)$$

(the system has solution for every parameter p), then the Herbrand's theorem gives us a list of terms $p_1(p), p_2(p), \ldots, p_n(p)$ (which are essentially polynomials with integer coefficients) such that a solution can be always found by trying all these values.

An example – T_{PV}

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- Reasonably axiomatized subsystem PV of T_{PV} is a well studied system of bounded arithmetic and can prove a lot of the contemporary complexity theory.

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- So we get that if

$$T_{\mathrm{PV}} \vdash (\forall x)(\exists y)\varphi(x, y),$$

then there exists $f \in L_{\mathrm{PV}}$ such that

$$T_{\mathrm{PV}} \vdash (\forall x)\varphi(x, f(x)).$$

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- A fundamental problem in complexity theory: Are any of $\mathbf{P}, \mathbf{NP}, \mathbf{coNP}$ equal? What about $\mathbf{P}$ and $\mathbf{NP} \cap \mathbf{coNP}$?

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Proof.

Let $\varphi(x)$ be of the form $(\exists y, |y| \leq p(|x|))(f(x,y) = 1)$, let $\psi(x)$ be of the form $(\forall y, |y| \leq q(|x|))(g(x,y) = 1)$, and let

$$T_{\mathrm{PV}} \vdash \varphi(x) \equiv \psi(x),$$

we also have

$$T_{\mathrm{PV}} \vdash \varphi(x) \vee \neg\psi(x).$$

By Herbrand's theorem we have that there exists a polynomial time h such that

$$T_{\mathrm{PV}} \vdash (\forall x)(f(x, h(x)) = 1 \vee g(x, h(x)) = 0)$$

now we can get a p-time algorithm deciding $\varphi(x)$ using f,g and h. □

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- KPT theorem: $\forall\exists\forall$ statement $\to$ a list of terms $t_1(a), t_2(a, b_1), \ldots, t_n(a, b_1, \ldots, b_{n-1})$, if the i-th term is not valid in a given model, it gives a value b_i (corresponding to the last $\forall$ quantifier) which can then be used to compute the next value. In any model, one of these terms is the witness.

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- This can be understood as a two player game, the teacher ($\forall$-player) and a student ($\exists$-player), the game is played in any model of the theory we are considering. The teacher always picks some element, the student tries to compute a potential witness using a term, and if the witness is wrong, the teacher provides a counter example, which the student can later use to find another potential witness.

Thank you for your attention!