A solution of Problem 6

Problem 6 in a sense incorporates all previous problems and its solution is, I think, the most subtle use of the idea to amplify the input/output ratio along a binary tree. It is a good idea to draw pictures of what is going on in the argument below.

Let us start with $g : x^2 \to x$ violating WPHP. Recall that we assume w.l.o.g. that x is a power of 2, $x = 2^k$, and that we think of x in a dual way: as set $\{0, \ldots, x-1\}$ and also as set $\{0,1\}^k$.

Consider a depth k binary tree, starting in a root and growing up: the root branches into two subtrees which are rooted at its left and right sons, resp., and they both have left and right sons themselves, etc. Imagine that we label the root of the tree by $u < x$. We extend the labeling to the whole tree by he following recursive process:

- *If $u \notin rng(g)$ label the sons of the root by $*$, their sons also by $*$, etc. all the way up to leaves.*
- *If $u = g(v, w)$ for some $v, w < x$ label the left son by v and the right son by w, and repeat the whole process for both subtrees.*
- *Stop when all leaves are labeled.*

It may be that only the root gets a label from x all other nodes get $*$ (this happens when $u \notin rng(g)$), or that only some nodes get a proper label and some get $*$.

Now we want to define the labeling in the theory and the problem is that the tree has $2x - 1 > 2^k$ nodes (inner nodes together with leaves) and hence a list of all labels would be a too long sequence: its length would be $> 2^k = x$ of numbers $< x$, i.e. it needs a code of size $> x^x$ and we cannot prove in the theory that such a number exists.

The key idea is that while we cannot define the whole labeling, we can define the label of any particular node. Think of $i = (i_1, \ldots, i_t) \in \{0,1\}^{\leq k}$ as a partial path in the binary tree, where 0 means go left and 1 means go right. The empty string i corresponds to the root, length one strings 0 and 1 correspond to the left and the right son of the root, resp., and strings of length k correspond to different leaves of the tree. We want to define in the theory the function:

$$(u)_i \ := \ \textit{the label of the node corresponding to } i \ .$$

As an example assume $i = (0, 1)$, i.e. we want to define the label of the right son of the left son of the original root. We have:

1. either $u \notin rng(g)$, or
2. $u = g(v_1, w_1)$ for some $v_1, w_1 < x$.

In case 1 we just define $(u)_i := *$. In case 2 we look at the the left son of the root, labeled by v_1, and consider two cases:

3. If $v_1 \notin rng(g)$ we define, as before, $(u)_i := *$.
4. If $v_1 \in rng(g)$ and $v_1 = g(v_2, w_2)$, we define $(u)_i := w_2$.

Note that in order to learn what the label of node i is we only needed to know labels of the nodes on the path i together with the labels of the nodes paired with them (the pairs are left and right sons of any node), all together 5 nodes (root labeled by u and the four nodes with labels v_1, w_1, v_2, w_2).

Function $(u)_i$ will be defined analogously for a general path $i = (i_1, \ldots, i_t) \in \{0,1\}^{\leq k}$: we need to know the labels on the path i ($t+1$ nodes when counting the root as well), and the labels of the nodes paired with them (another t nodes) certifying that all labels are in $rng(g)$. To summarize, the value $(u)_i$ is defined by a sentence of the form:

- *There exists a sequence s of length $2t+1$ of numbers $< x$, i.e. $s \leq x^{2t+1} \leq x^{2k+1} \leq x^{3|x|} \leq (x\#x)^3$, that lists labels on path i, starting with label u of the root, as well as labels of the nodes paired with the nodes on i, and the labels are assigned correctly as determined by map g.*

This solves the first part of Problem 6: to define $(u)_i$. Note that the function is definable by a bounded formula (with functions symbol $\#$).

For the second task let as assume we have Δ_0-definable $H \subseteq x (= \{0,1\}^k)$ and we want to find $u < x$ s.t. for all $i \in \{0,1\}^k$:

$$i \in H \ \text{ iff } \ (u)_i = 1 \ . \tag{1}$$

We shall find suitable u coding H in the sense of (1) by induction, by considering bigger and bigger pieces of H that are in the full binary tree above a certain inned node $j \in \{0,1\}^{\leq k}$. Let H_j denotes such a part of H: it concerns only those $i \in \{0,1\}^k$ that extend j.

If $j \in \{0,1\}^k$ then H_j is just one leaf (the j) and its code is simply its label, i.e. 1 if $j \in H_j$ and 0 otherwise. If $j \in \{0,1\}^{k-1}$, H_j are two neighboring leaves labeled by $v, w \in \{0,1\}$ (depending on their membership in H_j) with a common parent j , and the parent is labeled by $g(v,w)$. In general, when going from codes v for H_{j0} and w for H_{j1} take for the code of H_j the number $g(v,w)$.

The existence of a suitable u coding whole H is proved formally by induction as follows. For $\ell = 0, \ldots, k$ we prove:

- *For any $j \in \{0,1\}^{k-\ell}$ there is $u_j < x$ s.t. for any $i \in \{0,1\}^k$ extending j:*

  $$(u_j)_{(i-j)} = 1 \ \ \textit{iff} \ \ i \in H_j$$

  where $i - j$ is the path of length ℓ by which we need to extend j in order to get i, and H_j are the elements i of H that extend j.

Note that this can be written by a bounded formula because H is assumed to be Δ_0-definable.

Finally it remains to use this coding to prove in $I\Delta_0 + \Omega_1$ that all Δ_0-definable maps g satisfy $\mathrm{WPHP}_2^{x^2}$. As once before, we adopt Cantor's diagonal argument. Define set $H \subseteq \{0,1\}^k$ by:

$$i \in H \ \text{ iff } \ (i)_i = 0 \ .$$

The set is defined by a bounded formula hence it must be coded in the sense of (1) by some $u < x$. But then we get the contradiction as usual:

$$u \in H \Leftrightarrow (u)_u = 0 \Leftrightarrow u \notin H \ .$$

the first equivalence is by the definition of H, the second is by the choice of u as the coded of H. (Here we use the dual nature of x: $\{0, \ldots, x-1\}$ and $\{0,1\}^k$.)