Notes on Problems 2 and 3

Let us start with Problem 2. If $\varphi(x)$ violates IND:

$$\varphi(0) \wedge (\forall x < a, \varphi(x) \to \varphi(x+1)) \wedge \neg\varphi(a)$$

then, as in Folwarczny's solution of Problem 1,

$$I(y) \Leftrightarrow \forall x \leq y \varphi(x)$$

defines a cut below a, $I < a$. We know that we can define a shorter cut

$$J(x) \Leftrightarrow \forall y, I(y) \to I(y+x)$$

that is closed under addition. Because $I < a$ we can actually bound the quantifier by a:

$$J(x) \Leftrightarrow \forall y < a, I(y) \to I(y+x) \ .$$

All this is over PA^-, and J is Δ_0-definable because $\varphi \in \Delta_0$.

A simple idea would be to proceed as in Problem 1 and define a map sending $J < x < a$ to $\lfloor x/2 \rfloor$ and leaving $x \in J$ in place, but this is not 1-to-1. Note that it is 2-to-1 and some non-empty set of $\{0, \ldots, a-1\}$ (namely J) has 1 preimage, and this itself violates an obvious PHP-type principle. However, this is not of the form WPHP.

A less straightforward idea is to express any $x < a$ as a sum of powers of 2:

$$x = 2^{i_1} + \ldots + 2^{i_k} \ , \quad \text{with} \ \ i_1 > \ldots > i_k \tag{1}$$

and then move each power $2^i \notin J$ to 2^{i-1}, while leaving the powers that are in J in place. Note that, because J is closed under addition, this is 1-to-1 and maps a to $a/2$ (assume w.l.o.g. that a itself is a power of 2).

Formally proceed as follows:

1. Define predicate $Pow(y)$: *y is a power of* 2, by saying that each proper divisor is even.

2. Define relation $P(x, y)$: *y is a power of* 2 *and it occurs in the unique expression (1) for x*:

   $$Pow(y) \wedge \ \exists u, v \leq x, \ u + y + v = x \wedge 2y | u \wedge v < y \ .$$

   Note that there are $\leq |x|$ elements y satisfying $P(x, y)$.

3. Define relation $Q(x, z)$:

   $$\exists y \leq x, \ P(x, y) \wedge [(y \in J \wedge y = z) \vee (y \notin J \wedge y = 2z)] \ .$$

4. Then define map f:

   $$f(x) \ := \ \sum \{z < x \mid Q(x, z)\} \ .$$

Note that such f is 1-to-1 and its range is $\leq a/2$. But the issue is how do we know that it is well defined?

In order to prove that the sum exists we can proceed by induction on t to show that

$$\exists w \leq x,\ w = \sum \{z < x \mid Q(x, z)\} \cap \{0, \ldots, t\}$$

and then conclude that $f(x)$ is defined by taking $t = |x|$. But we are supposed to work over PA^- and we assume a failure of IND, so this cannot be done.

While you will be thinking how to possibly overcome this - I do not know - try also:

- Write down the definition of the sum formally: there exists a sequence ...
- Extend the idea to WPHP from n^2 to n.
- Problem 3: verify that all manipulations so far did not depend on the language, i.e. it is OK to have a new relation symbol.