Topological view of some problems in complexity theory

- infinitary circuits

- Borel sets vs. circuit-definable sets

- Borel hierarchy vs. expressive power of circuits

- analytic sets vs. non-deterministic circuits

- complements of analytic sets (are not necessary analytic)

Topology

$\mathcal{E}$ is defined to be $\{0, 1\}$.
Note that $\mathcal{E}$ has trivial discrete topology
(every subset of $\mathcal{E}$ is considered to be open).

We want to introduce topology on $\mathcal{E}^\omega$
(set of infinite sequences of 0 and 1).
Elements of $\mathcal{E}^\omega$ are called "reals" $\left(\begin{matrix} (010011\ldots\ldots) \\ (1111\ldots\ldots) \end{matrix} \right)$

Prop. There is a 1-1 correspondence between "reals" and reals (elements of $\mathbb{R}$)
Pr: simple cardinality argument or not so simple explicit construction ∎

From now on, when we will mention reals, we will mean elements of $\mathcal{E}^\omega$.

So what about the topology?
We may use the normal $\mathbb{R}$-topology, but it is not useful.

Let us use the fact that $\mathcal{E}^\omega$ is the product of $\mathcal{E}$, which have dicrete topology.

Topology of Σ^ω

Step 1: for every $n \in \mathbb{N}$ and $a \in \{0,1\}$ we define the set P_n^a as a set of all sequences of Σ^ω s.t. their n-th element (bit) is a (pre-base sets)

Ex: $P_0^0 = \{(0, \dots\dots)\}$, $P_0^1 = \{(1, \dots\dots)\}$.

Step 2: we define base sets as finite intersections of sets defined in the 1st step.

Ex: $P_0^0 \cap P_1^0 \cap P_2^0 = \{(0,0,0,\dots)\}$.

Step 3: we define open sets as unions of base sets (possibly infinite unions)

! it suffices to have only countable unions.

Prop.: sets defined in the 3rd step form a topology on Σ^ω.

Pr: just simply check axioms or use the fact, that this is an instance of more general product topology.

E.g.: $\emptyset = P_0^0 \cap P_0^1$, $\Sigma^\omega = P_0^1 \cup P_0^0$. Unions are trivial, intersections are more interesting ... ■

Infinite circuits

Finite circuits (with n-variables) can define subsets of $\mathcal{E}^n$.

Infinite circuits define subsets of $\mathcal{E}^\omega$.

Let us proceed slowly and create circuits that will correspond to our topology.

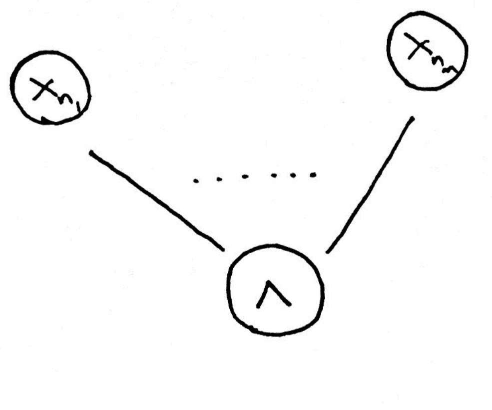
What circuits can define open sets?

1. How to define P_n^a-set for fixed a and n?
 Easy! Just take x_n, if a is 1, or $\neg$ — x_n, otherwise.

We will use $\overline{x_n}$ instead of the latter.

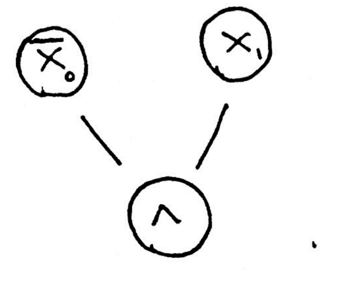
2. How to define base set? Let $B := P_{n_1}^{a_{n_1}} \cap \ldots \cap P_{n_m}^{a_{n_m}}$.

Then we can use the circuit:



, where each variable is taken as itself or with an overline, according to a_{n_i}.

<u>Ex</u>: $P_0^0 \cap P_1^1 \rightsquigarrow$

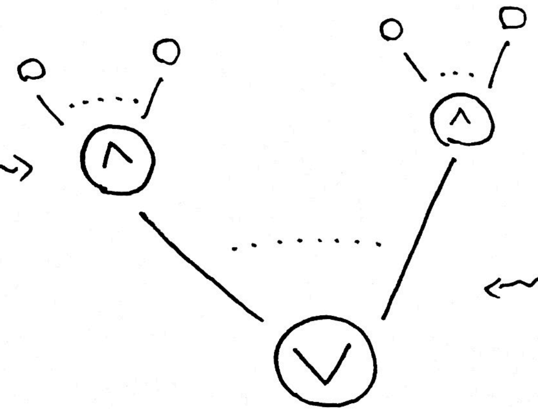


.

Infinite circuits
(cont.)

3. How to define open sets? ... $\underbrace{O}_{\text{some set}} := \bigcup_{n\in\mathbb{N}} \overbrace{B_n}^{\text{base sets}}$

circuits, defining B_n sets.



Finitely many edges

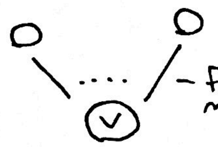
countably many edges

Let us introduce formal definitions:

$\boxed{\text{Def:}}$ Π_0-circuit / Σ_0 is <u>finite</u> set of variables (or negated variables).

For $i>0$: Σ_i-circuit is countable collection of Π_{i-1}-circuits. Analogously Π_i.

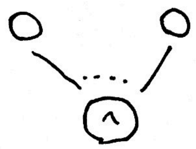
<u>Ex</u>: $\Sigma_0 \rightsquigarrow$



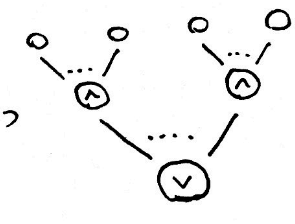
- variables

- finitely many edges

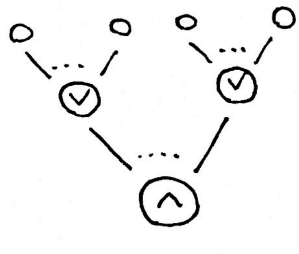
$\Pi_0 \rightsquigarrow$



$\Sigma_1 \rightsquigarrow$



$\Pi_1 \rightsquigarrow$

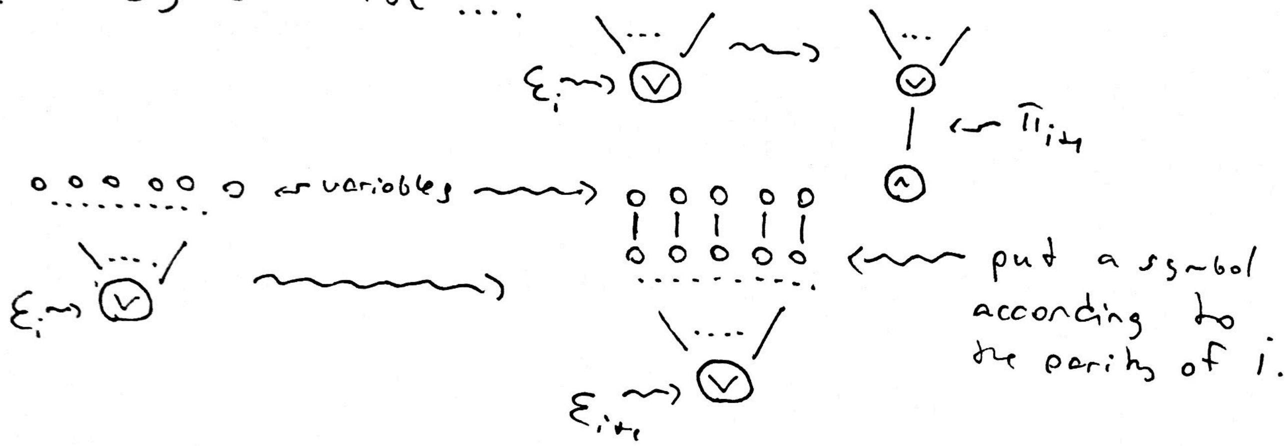


$\boxed{\text{Def:}}$ we say that a circuit C <u>accepts</u> a real α iff ..... naturally but cumbersome.

Borel sets

Prop.: $\Sigma_i \subseteq \Sigma_{i+1}$, $\Sigma_i \subseteq \Pi_{i+1}$, $\Pi_i \subseteq \Pi_{i+1}$, $\Pi_i \subseteq \Sigma_{i+1}$ in the sense of set definability (i.e. everything definable by Σ_i-circuit is also definable by Σ_{i+1} circuits).

Pr: easy exercise ....



∎

Prop: open subsets of Σ^ω are exactly Σ_1-definable.

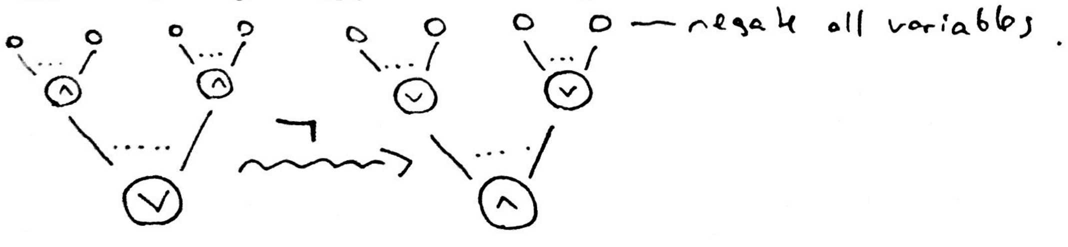
Pr: we have already showed that open sets can be Σ_1-definable.

To prove the other implication, note that Π_0 ~~set~~ circuits define base sets and $\vee$ is behaving as a countable union operator

∎

Prop.: closed sets are exactly Π_1-definable.

Pr: as above. or using de Morgan laws:



Borel sets

$\boxed{\text{Def:}}$ **Borel sets** is the smallest (via $\subseteq$) system of subsets of $\mathcal{E}^{\omega}$ (reals), containing open sets and closed under (complements) and countable unions {and intersections}.

Elegant definition but does not tell us much about their appearance.

$\boxed{\text{Def:}}$ **Σ_1-Borel** are open sets, **Π_1-Borel** are closed sets.

Σ_i-Borel is the system containing countable unions of Π_{i-1}-Borel sets. Analogously for **Π_i-Borel**.

Prop: $\Sigma_i\text{-Borel} \subseteq \Sigma_{i+1}\text{-Borel}$, $\Pi_i\text{-Borel} \subseteq \Pi_{i+1}\text{-Borel}$

$\Sigma_i\text{-Borel} \subseteq \Pi_{i+1}\text{-Borel}$, $\Pi_i\text{-Borel} \subseteq \Sigma_{i+1}\text{-Borel}$.

Π_i-Borel are complements of Σ_i-Borel.

Pr: First part is trivial. And the second, too.

■

This construction can be continued transfinitely up to ω_1. This way we will get all Borel sets.

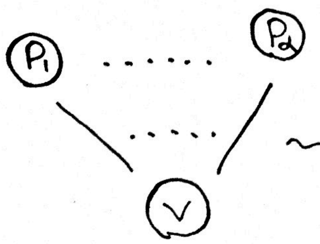
$$\text{Borel} = \bigcup_{i<\omega_1} \Sigma_i\text{-Borel} = \bigcup_{i<\omega_1} \Pi_i\text{-Borel}.$$

Borel vs. circuits

Thm.: Σ_i-Borel sets are exactly Σ_i-definable.
Π_i-Borel sets are exactly Π_i-definable.

Pr: These two claims are done simultaneously by induction. We have already showed the base case.

Assume S is Σ_{i+1}-Borel. Then it is countable union of Π_i-Borel sets P_α, which are Π_i-definable by circuits P_α. Then create a circuit:



$\sim$ Σ_{i+1}-circuit.

The same goes for Π_{i+1}.

To prove the other implication, note that Σ_{i+1}-definable is countable union of Π_i-definable ... ■

It is well-known fact, that Borel hierarchy is strict, i.e. $\Sigma_i \subsetneq \Sigma_{i+1}$. (This holds for transfinite elements, too).

That means, that our Σ-circuits have no total control over Borel sets, and Σ_{i+1}-circuits have more expressive power than Σ_i-circuits.

But topological proof uses diagonal argument and does not really show us some circuit constructions.

Expressive power

There is an explicit function $F_d : \Sigma^\omega \to \Sigma$, s.t. it is Σ_d-definable but not Σ_{d-1}.

Recall $\langle \, \rangle_d : \mathbb{N}^d \to \mathbb{N}$ ... pairing function

We can pick any fixed bijection we want but let us stick to $\langle \, \rangle$ s.t. $\langle a, b \rangle_2 := \frac{(a+b)(a+b+1)}{2} + b$.

We can proceed inductively to define

$$\langle a_1, \dots, a_d \rangle_d := \langle a_1, \langle a_2, \dots, a_d \rangle_{d-1} \rangle_2 .$$

We then define F_d as $F_d(\alpha) = \bot$, iff

$$\exists i_1 \forall i_2 \exists i_3 \dots Q i_d \left(\alpha(\langle i_1, \dots, i_d \rangle) = 1 \right),$$

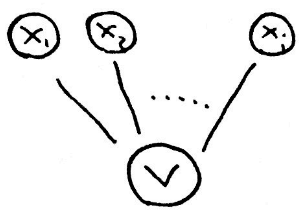
where Q is $\exists$ or $\forall$ according to parity of d, and $\alpha(\langle i_1, \dots, i_d \rangle)$ is an element of α on the position $\langle i_1, \dots, i_d \rangle \in \mathbb{N}$.

Ex: $F_1(\alpha) = \bot$ iff $\exists i \left(\alpha(i) = 1 \right)$

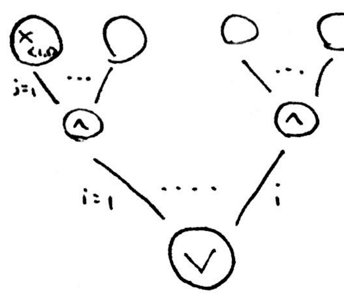
$F_2(\alpha) = \bot$ iff $\exists i \forall j \left(\alpha\left(\frac{(i+j)(i+j+1)}{2} + j \right) = 1 \right)$.

Prop: F_d is Σ_d-definable. (but not Σ_{d-1}-definable)

Pr: by induction. F_1 is definable by



F_2 is definable by



$\bigcirc$ - variables, $x_{\langle i, j \rangle}$.

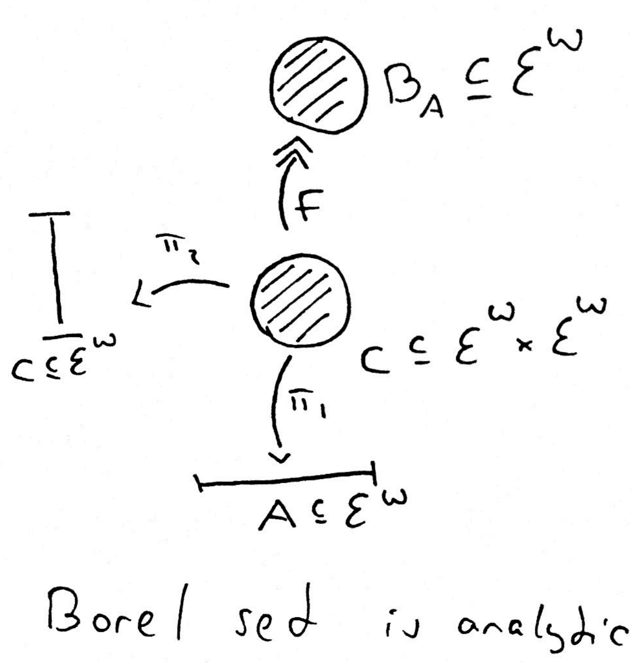
Analytic sets

Let f be bijective map between $\mathcal{E}^\omega \times \mathcal{E}^\omega$ and $\mathcal{E}^\omega$.
We can pick any possible, but let us stick to the simplest one :

$$\begin{matrix} 1\,1\,1 \ldots\ldots \\ 0\,0\,0 \ldots\ldots \end{matrix} \xrightarrow{f} 101010\ldots\ldots$$

$\boxed{\text{Def:}}$ Set A is called <u>analytic</u>, if there exist some Borel set B_A s.t.

$$A = \{\alpha \mid \exists \beta \text{ s.t. } f(\alpha, \beta) \in B_A\}.$$



<u>Thm</u> every Borel set is analytic.

<u>Pr</u>: it suffices to show that ~~are~~ base sets are analytic, and that analytic sets are closed under countable unions and intersections.

Why P_n^a are analytic? Consider P_0^0 ....

$$P_0^0 = F(P_0^0 \times \mathcal{E}^\omega) \quad \ldots \begin{matrix}(0,\ldots\ldots) \\ (\ldots\ldots)\end{matrix} \xrightarrow{f} (0,\ldots\ldots), \quad \underset{\overset{\|}{P_0^0}}{\pi_1(P_0^0 \times \mathcal{E}^\omega)}.$$

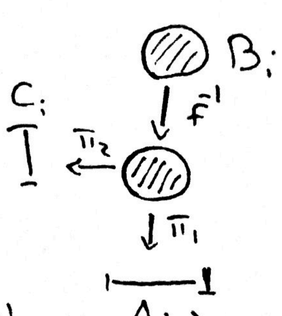
Analogously for general a and n.

Analytic sets
(cont.)

Now, assume $A = \bigcup_{i \in \mathbb{N}} A_i$, where A_i - analytic.

That means that $A_i = \pi_1\left(f^{-1}(B_i)\right)$

projection — some Borel set.



So, $A = \bigcup_{i \in \mathbb{N}} \pi_1(f^{-1}(B_i)) = \pi_1\left(\bigcup_{i \in \mathbb{N}} f^{-1}(B_i)\right) = \pi_1\left(f^{-1}\left(\bigcup_{i \in \mathbb{N}} B_i\right)\right)$,

but $\bigcup_{i \in \mathbb{N}} B_i$ is Borel set, which proves analyticity of A.

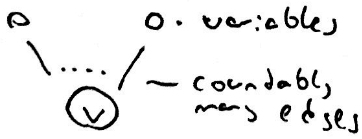
The same goes for countable intersections ■

Analytic sets are not closed under complements!
(which means that not all analytic sets are Borel).

What is the correspondence between analytic sets and circuits?

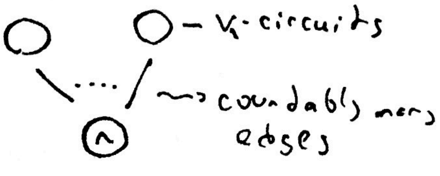
We want to introduce new type of infinitary circuits.

$\boxed{\text{Def:}}$ We call a circuit V_1, if it is of the form



(e.g. it can be viewed as Σ_2-circuit)

A circuit is called Λ_2, if it is of the form



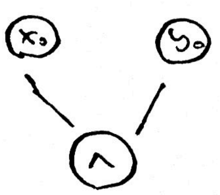
(Π_3-circuits).

Non-deterministic circuits

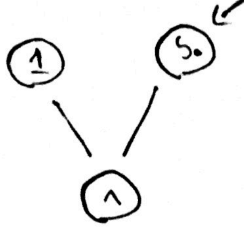
Def: We call a $\wedge_2$-circuit a <u>non-deterministic</u>, if it contains some variable from the set $\{y_0, \bar{y}_0, y_1, \bar{y}_1, \dots\}$, disjoint with the set of input variables.

We say, that ND-circuit accepts real α, if there exist some setting of non-determinate variables, s.t. this circuits accepts α with this setting according to the standard rules.

<u>Ex</u>:



$\rightsquigarrow$ accepts any ~~set~~ real from P_0^{ω}



put $y_0 = \underline{1}$.

does not accept any element of P_0^0.

<u>Thm</u> ND-circuits accept exactly analytic sets.

<u>Pr</u>: (sketch)

Let us show how to create an ND-circuit, accepting a Borel set B. Assume, for the sake of simplicity, that B is Σ_3-Borel, i.e. it is a countable union of sets A_i, where each A_i is a countable intersection of C_j^i, where C_j^i are open.

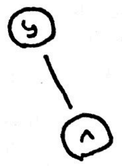
$$B = \bigcup_{i \in \mathbb{N}} \left(\bigcap_{j \in \mathbb{N}} C_j^i \right), \quad C_j^i \cdot \text{open}.$$

It is obvious that we cannot hope to be able to form a circuit without a use of indeterminate variables. How can we use them?

ND-circuits
(cont.)

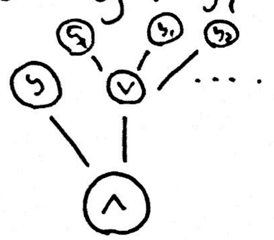
We want to somehow ~~m.~~ simulate the sets which are used in the construction of B.

So y corresponds to the B itself:



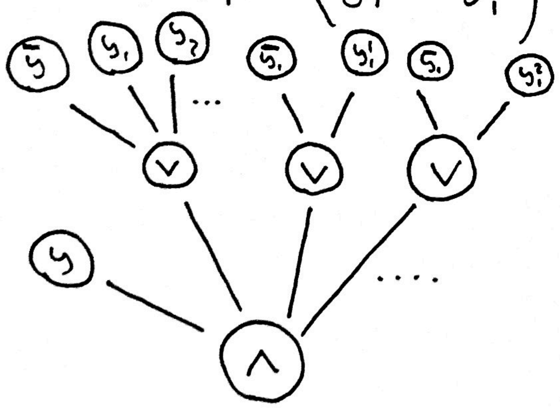
. Then y_i correspond to A_i.

Because B is a union of A_i, y "forces" at least one of y_i. $y \to (y_1 \vee y_2 \vee \dots)$, which can be rewritten to $\bar{y} \vee y_1 \vee y_2 \vee \dots$.



Let y_i^j correspond to C_i^j.

Because A_i is an intersection of C_i^j, y_i "forces" all other y_i^j. $(y_i \to y_i^1) \wedge (y_i \to y_i^2) \wedge \dots$.

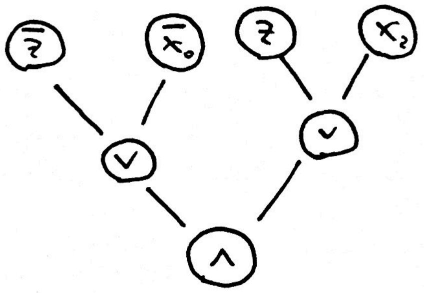


Each C_i^j is a union of base sets, say B_i^{jK}. So we similarly add new indeterminate variables and proceed as above. Finally, how will we introduce determinate variables?

ND-circuits (cont.)

Assume some indeterminate variable z corresponds to some base set, for the sake of simplicity let this base set be of the form $\{(0,..,1,....)\}$, i.e. $P_0^0 \cap P_2^1$.

This z forces the first bit to be 0, and the third to be 1, which corresponds to the following configuration: $(z \to \overline{x}_0) \wedge (z \to x_2)$.



It is not hard to see that the circuit we get is really Λ_2. ■

To prove the second implication, assume our ND-circuits accepts a set B. We can also imagine our ND-circuit as a normal Π_3-determinate circuit. But it then accepts a Borel set A, and our B is just a projection of A, which shows analitycity

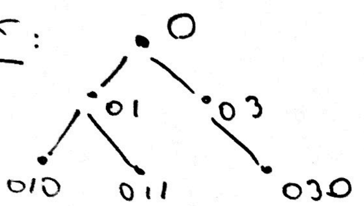
■

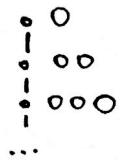
Complements

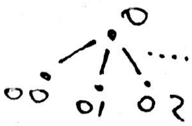
We want to show that analytic sets are not closed under complements using combinatorial arguments.

Def: Let $\mathbb{N}^*$ be the set of all finite sequences of $\mathbb{N}$. A <u>tree</u> is a subset of $\mathbb{N}^*$, closed under prefix.

Ex:







The set $\mathbb{N}^*$ is countable. Fixing some enumeration of $\mathbb{N}^*$, we get the correspondence between trees and reals. So we can ~~say~~ speak about Borel or analytic sets of trees.

Def: A tree is called <u>well-founded</u> if it has no infinite branch (but it can be of infinite size).

Ex:

$\rightsquigarrow$ is well-founded and infinite.

Let W be the set of all well-founded trees and $\overline{W}$ - its complement (trees, which have infinite branches).

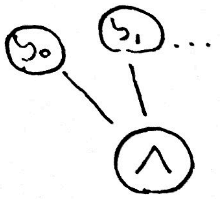
We will show that $\overline{W}$ is analytic, but W is not.

To prove the first part, we will use the theorem, which characterizes analytic sets, as ND-definable.

W vs. $\overline{W}$

Thm.: $\overline{W}$ is analytic, i.e. there is a Λ_2-ND circuit* which accepts exactly $\overline{W}$.

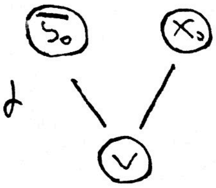
Pr: The idea is to "guess" an infinite branch non-deterministically.
Such infinite branch is simulated via non-deterministic variables $y_0, y_1, \dots$.



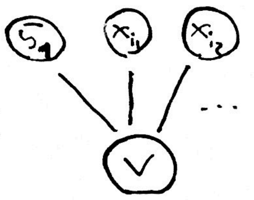
← this tells us that there should be some infinite branch.

y_0 should force our tree to have some node on 0-th level, y_1 forces nodes on 1-st level and so on.

This shows that we should also add, for example $y_0 \to x_0$, which is represented via $\vee_1$-circuit ($y_0 \to x_0 \Leftrightarrow \neg y_0 \vee x_0$)



y_1 forces us to pick something on the first level, but there are countably many options, which still can be represented via $\vee_1$-circuit:



x_{i_k} - correspond to all possible nodes on the 1-st level.

We proceed the same way with every y_n.

It is not hard to show that such ND-circuits defines $\overline{W}$, meaning it is analytic. ∎

So let us now show, that W is not.

W vs. $\dot{\overline{W}}$ (cont.)

For $s, t \in \mathbb{N}^*$ we denote st as their concatenation.

Ex.: $(0,0)(1,0) = (0,0,1,0)$

For $s \in \mathbb{N}^*$ and $A \subseteq \mathbb{N}^*$ we denote sA as $\{st \mid t \in A\}$.

Def.: For any tree τ and $s \in \mathbb{N}^*$, we define the <u>detail of τ at s</u> as $\tau^s := \{t \mid st \in \tau\}$.

Ex.: for $s = (11)$ and $\tau =$

The <u>detail of A at s</u> (where $A \subseteq \mathbb{N}^*$) is a set $\{\tau^s \mid \tau \in A\} =: A^s$.

We say that $A \subseteq \mathbb{N}^*$ <u>is large at s</u>, if $W \subseteq A^s$, or simply <u>large</u>, if it is large at some $s \in \mathbb{N}^*$.

Ex.: sW is large at s. W is large at e (at any fixed s)

The set $\Big\{$  $\Big|$ ω is an element of $01W \Big\}$ is large at 01.

Now we prove some Ramsey-like property of systems of trees.

Combinatorial properties

Prop. Assume $A = B_1 \cup B_2 \cup \dots$ - countable union of sets of trees s.t. A is large at $s \in \mathbb{N}^*$. Then, for some $i, j \in \mathbb{N}$, B_i is large at sj.

sequence of length 1

Pr: Note that the addition of j after s is necessary, for example, let $B_i = s\,i\,W$. Then, A is large at s, but no B_i is large at s.

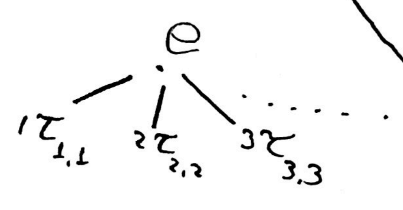
So, assume this does not hold, i.e. for each i, j B_i is not large at sj. So each detail of B_i at any sj lacks some tree $\tau_{i,j}$ in W.

Let us create a tree $\sigma := 1\tau_{1,1} \cup 2\tau_{2,2} \cup \dots$.

σ is definitely well-founded.

But σ is not in B_i^s for any i.

That means that σ is not in A^s, which contradicts the largeness of A at s.

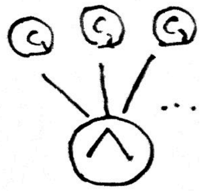


$$A^s = \Big(\bigcup_{i \in \mathbb{N}} B_i\Big)^s = \bigcup_{i \in \mathbb{N}} B_i^s$$

Arriving at a contradiction

Thm. W is not analytic.

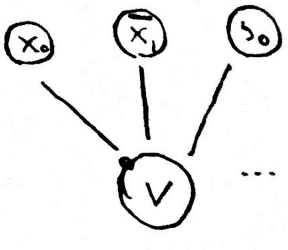
Pr: Assume W is analytic, accepted by a non-deterministic circuit N:



$\rightsquigarrow$ denote $\vee$-circuits of N as $C_1, C_2, \dots$

Each tree from W is accepted by N via some configuration of non-deterministic variables $y_0, y_1, \dots$. For a $\omega \in W$ we denote its <u>proof</u> (π_ω) as two sequences of bits, the first is ω itself and the second is corresponding setting of non-deterministic values (we obviously may have more than one proof).

We say that C_i-circuit ($\vee$-circuit) <u>accepts</u> the proof π_ω at some index $j \in \mathbb{N}$, if j-th literal (x_j or $\overline{x_j}$ or y_j or $\overline{y_j}$) is the "reason", why this proof is accepted by C_i (in other words, j-th literal of C_i becomes 1, after plugging-in the proof π_ω).



... for a $\omega = (0, 0, \dots)$, this circuit accepts the proof $\pi_\omega = ((0,0,\dots),(1,\dots))$ at 1 or 0

for det. variable

for indet. variable

The idea of proof is to construct a sequence $W \supseteq A_1 \supseteq A_2 \supseteq \dots$ s.t. there will be exactly one tree $\alpha \in \bigcap_{i \in \mathbb{N}} A_i$, which will be accepted by N and also contain an infinite branch, which will be our contradiction.

Arriving at a contradiction (cont.)

The construction is done inductively.
On each stage i, we will set $A_i \subseteq A_{i+1}$ and also some $s_i \in \mathbb{N}^*$ (tree node) and p_i - sequence of natural numbers of length i.

Stage 0: Let $A_0 := W$, $s_0 := e$ and $p_0 := e$.
Note that A_0 is large at s_0.

Stage i+1: Let B_m be the set of all trees of A_i, s.t. they have proofs accepted by C_i at m. Obviously, $A_i = \bigcup_{m \in \mathbb{N}} B_m$.
Using Ramsey-like property of systems of trees we can find some m, n s.t. B_m is large at $s_i n$

from the previous step. We assume A_i is large at s_i.

So let us fix some m, n.
We can also assume that all elements of $\mathbb{N}^*$ (nodes) all enumerated, meaning $\mathbb{N}^* = \{t_1, t_2, \dots\}$.
So let D be $\{d \in B_m \mid d \text{ contains } t_{i+1}\}$.

as a node

Obviously, $B_m = D \cup \overline{D}$, so, again by using our Ramsey-like property, we can

(relative to B_m)

deduce that either D or $\overline{D}$ is large at a sequence $s_i n k$. Let A_{i+1} be this large set.

Arriving at a contradiction (end)

Let then S_{i+1} be $S_i n k$ and P_{i+1} be $P_i m$.

What can we say about A_{i+1}?

- A_{i+1} is large at S_{i+1}.

(*) all trees of A_{i+1} either have t_{i+1} as their node or not. Assuming this property is preserved under inductive operation, we get that all elements of A_{i+1} agree on nodes $\{t_1, t_2, \dots, t_{i+1}\}$ (each such node is either an element of all trees or does not belong to any of them).

(**) all trees of A_{i+1} are well-founded and have proofs which are accepted by C_{i+1} at m. Assuming this property is preserved and noting that the sequence P_{i+1} has length $i+1$, we can see that each tree of A_{i+1} has a proof which is accepted by C_j at $P_{i+1}(j)$ for $j \leq i+1$.

j-th position of the sequence P_{i+1}.

Using (*) we see that there is only one tree α which belongs to $\bigcap_{i \in \mathbb{N}} A_i$.

This tree has an infinite branch:

$e := s_0$

$e n_0$

$e n_0 k_0 =: s_1$

$s_1 n_1$

$s_1 n_1 k_1 =: s_2$

...

But this tree is also accepted by N via the proof constructed according to the infinite sequence $P := P_1 \cup P_2 \cup \dots$.

we extend our finite sequences

We take $p(j)$-th indeterminate variable $y_{p(j)}$ as 1 iff j-th inductive step "says so"