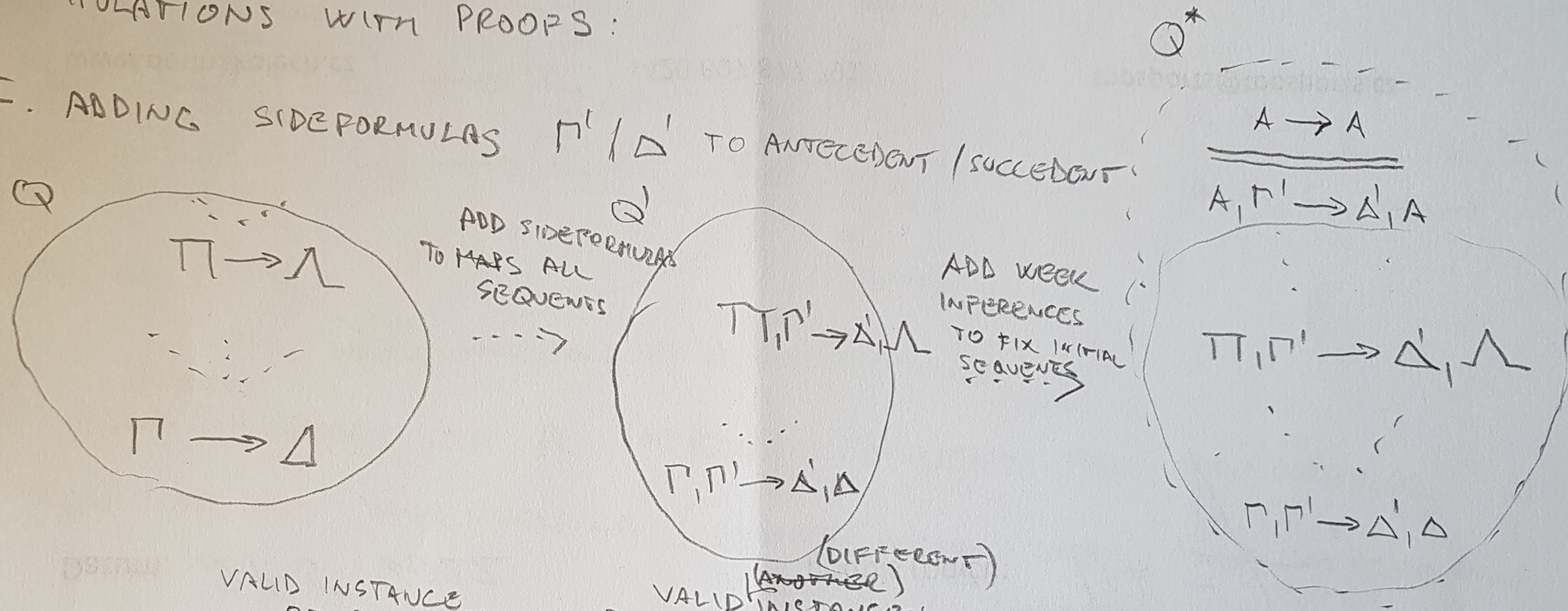
MANIPULATIONS WITH PROOFS:

I. ADDING SIDEFORMULAS Γ' / Δ' TO ANTECEDENT / SUCCEDENT:



VALID INSTANCE OF AN INFERENCE RULE I $\Rightarrow$ VALID INSTANCE OF ~~AN~~ (~~ANOTHER~~) (DIFFERENT) THE SAME INFERENCE RULE I

INFERENCES: (ONE UPPER SEQ)

$$\frac{\Pi \rightarrow \Lambda}{\Sigma \rightarrow \Phi} \dashrightarrow \frac{\Pi, \Gamma' \rightarrow \Delta', \Lambda}{\Sigma, \Gamma' \rightarrow \Delta', \Phi}$$

(TWO UPPER SEQS)

$$\frac{\Pi_1 \rightarrow \Lambda_1 \quad \Pi_2 \rightarrow \Lambda_2}{\Sigma \rightarrow \Phi} \dashrightarrow$$

INITIAL SEQUNTS: (A AN ATOMIC FLA)

$$A \rightarrow A \dashrightarrow A, \Gamma' \rightarrow \Delta', A \dashrightarrow \frac{A \rightarrow A}{A, \Gamma' \rightarrow \Delta', A}$$

$\|Q\| = \|Q'\| = \|Q^*\|$, Q, Q', Q^* HAVE "THE SAME" STRONG INFERENCES ~~POSSIBLY~~ WITH ~~THESE~~ ~~the~~ DIFFERENT & EXCEPT ~~UP TO~~ SIDEFORMULAS EXCEPT FOR

II. REMOVING <u>DIRECT</u> ANCESTORS OF A GIVEN <u>NON-ATOMIC</u> FORMULA IN THE ENDSEQUENT (ANTECEDENT / SUCCEDENT)

A A A (OCCURRENCE OF)

Q

$$\Pi \to \Lambda$$
$$\vdots$$
$$\Pi \to \Delta$$

$\dashrightarrow$

Q'

$$\Pi^- \to \Lambda$$
$$\vdots$$
$$\Pi^- \to \Delta$$

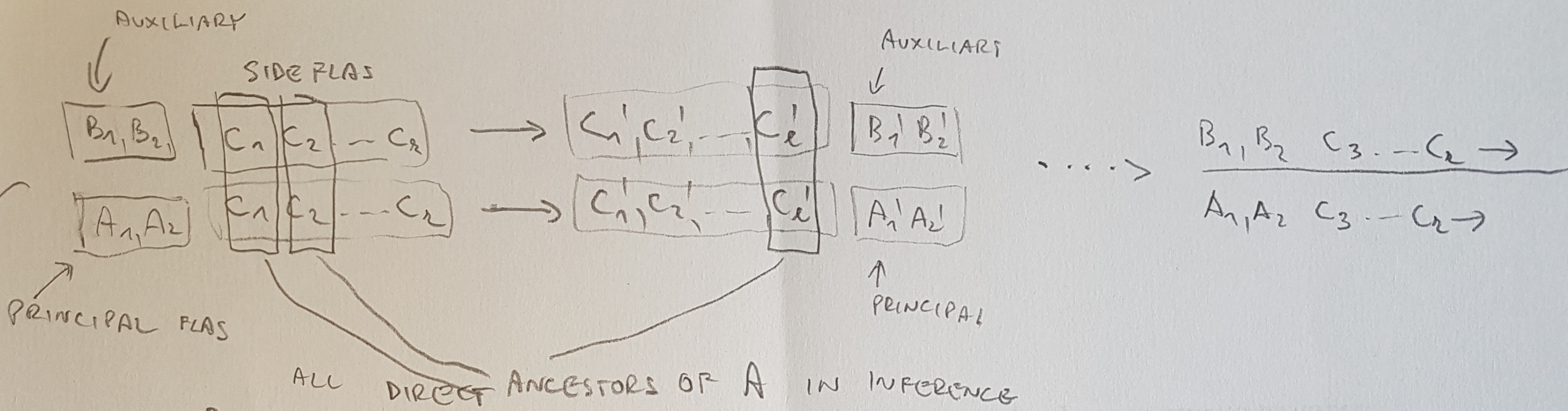
INITIAL SEQUENTS: (D ATOMIC)

$D \to D \quad \dashrightarrow \quad D \to D$

$D \neq A$ (ATOMIC / NON-ATOMIC)

INFERENCES:

$$\frac{\Pi \to \Lambda}{\Sigma \to \Phi} \quad \dashrightarrow \quad \frac{\Pi^- \to \Lambda}{\Sigma^- \to \Phi}$$



- IF ALL DIRECT ANCESTORS OF A IN LOWER SEQUENT ARE SIDE FORMULAS, THEN WE GET VALID INFERENCE (ANOTHER INSTANCE OF THE SAME INFERENCE RULE WITH LESS SIDE FORMULAS)
- IF DIRECT ANCESTOR OF A IS A PRINCIPAL FORMULA OF A STRUCTURAL RULE, THE INFERENCE CAN BE REMOVED

I.E.

EXCHANGE : LEFT $\quad \dfrac{\Gamma, \boxed{A}, B, \Pi \to \Delta}{\Gamma, B, \boxed{A}, \Pi \to \Delta} \quad \dots\to \quad \dfrac{\Gamma^-, B, \Pi^+ \to \Delta^+}{\Gamma^-, B, \Pi^- \to \Delta^-}$

CONTRACTION : RIGHT $\quad \dfrac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A} \quad \dots\to \quad \dfrac{\Gamma^- \to \Delta^-}{\Gamma^- \to \Delta^-}$

FOR A NONATOMIC,

- THERE IS EXACTLY ONE LOGICAL INFERENCE RULE I S.T.

  A CAN BE THE PRINCIPAL FORMULA IN AN INSTANCE OF I

  DETERMINED BY THE OUTERMOST LOGICAL CONNECTIVE OF A AND

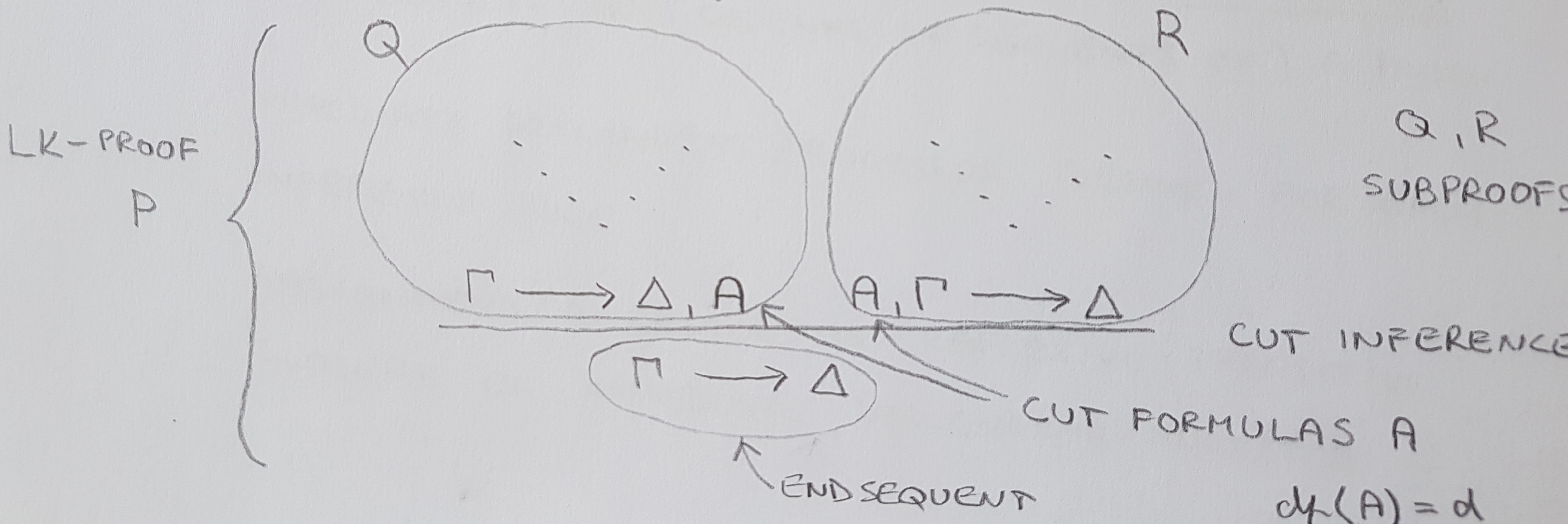
  ~~ITS OCURENCE BY THE S~~ ON BY THE ~~ITS~~ OCCURENCE OF A ON LEFT/ RIGHT OF →

- FOR A ATOMIC, THERE IS NO LOGICAL INFERENCE RULE I S.T

  A CAN BE THE PRINCIPAL FLA IN AN INSTANCE OF I

LEMMA: LET P BE AN LK-PROOF WITH FINAL INFERENCE A CUT OF DEPTH d SUCH THAT EVERY OTHER CUT IN P HAS DEPTH STRICTLY LESS THEN d.

THEN THERE IS AN LK-PROOF P^* WITH THE SAME ENDSEQUENT OF P WITH ALL CUTS IN P^* OF DEPTH $< d$ AND WITH $\|P^*\| < \|P\|^2$.



PROOF OF THE LEMMA :

BY CASES ON THE OUTERMOST LOGICAL CONNECTIVE OF THE CUT FORMULA A

$$\dfrac{\overset{Q}{\Gamma \to \Delta, A} \qquad \overset{R}{A, \Gamma \to \Delta}}{\Gamma \to \Delta}$$

- BASE CASE : A ATOMIC
- CASES : $\neg, \wedge, \vee, \supset, (\forall x), (\exists x)$

- WE ASSUME P TO BE IN FREE VARIABLE NORMAL FORM
- WE CAN ASSUME, THAT BOTH BRANCHES OF THE PROOF Q, R CONTAIN AT LEAST ONE STRONG INFERENCE RULE (I.E. $\|Q\|, \|R\| \geq 1$)

SUPPOSE $\|Q\| = 0$. THEN Γ CONTAINS A, OR $\exists B : \Gamma, \Delta$ CONTAINS B

$$\dfrac{R}{\Gamma \to \Delta}\ \text{WEAK} \qquad\qquad \dfrac{B \to B}{\Gamma \to \Delta}\ \text{WEAK}$$

BASE CASE : <u>A ATOMIC :</u>

Q R

$$P: \quad \frac{\Gamma \longrightarrow \Delta, A \qquad A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

$dp(A) = 0 \Rightarrow$ NO CUTS IN Q, R

ADD Γ TO ANTEC. Δ TO SUCCED. — REMOVE DIRECT ANCESTORS OF A — FIX INITIAL SEQUENTS

P^* : $Q \cdots\!\!> Q' \cdots\!\!> Q'' \cdots\!\!> P^*$

IN Q: $B \longrightarrow B$ ... IN Q': $B, \Gamma \longrightarrow \Delta, B$ ... IN Q'': ... IN P^*:

$B = A$ IS A DIR ANCESTOR: $B, \Gamma \longrightarrow \Delta, B \cdots\!\!> A, \Gamma \longrightarrow \Delta \cdots\!\!> R$

OTHERWISE: $B, \Gamma \longrightarrow \Delta, B \cdots\!\!> \dfrac{B \longrightarrow B}{B, \Gamma \longrightarrow \Delta, B}$ WEAK

$\|Q\| = \|Q'\| = \|Q''\|$

$\|P^*\| \leq \|Q\| + (\|Q\|+1)\|R\| < (\|Q\|+1)(\|R\|+1) < \|P\|^2$, NO CUTS IN P^*

↳ # OF INITIAL SEQUENTS $\leq$ # STRONG RULES + 1

BY INDUCTION : ONLY INITIAL SEQUENT ✓

"UNARY" INFERENCE RULE ✓

"BINARY" INFERENCE RULE $\longrightarrow$ STRONG INFERENCE RULE

$I = I_1 + I_2$ $\quad$ $I_1 \leq S_1 + 1$ $\quad$ $I_2 \leq S_2 + 1$ $\quad$ $S = S_1 + S_2 + 1$

$I_1 + I_2 \leq S_1 + S_2 + 2 = (S_1 + S_2 + 1) + 1$

CASE $A = \neg B$: IDEA:

$$\overbrace{\dfrac{\underset{Q}{\Gamma \longrightarrow \Delta, \neg B} \qquad \underset{R}{\neg B, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta}}^{P} \text{ CUT } dp(B)+1 \quad \cdots\rightarrow \quad \overbrace{\dfrac{\underset{Q^*}{B, \Gamma \longrightarrow \Delta} \qquad \underset{R^*}{\Gamma \longrightarrow \Delta, B}}{\Gamma \longrightarrow \Delta}}^{P^*} \text{ CUT } dp(B)$$

$$\underset{\text{VALID}}{Q} \xrightarrow{\text{ADD } B \text{ TO ANTECEDENT}} \underset{\text{INVALID}}{Q'} \xrightarrow{\text{FIX INITIAL SEQUENTS}} \underset{\text{VALID}}{\tilde{Q}} \xrightarrow{\text{REMOVE DIRECT ANCESTORS OF } \neg B} \underset{\text{INVALID}}{Q''} \xrightarrow{\text{FIX } \neg\text{:RIGHT INFERENCE}} \underset{\text{VALID}}{\hat{Q}}$$

IN Q: IN $Q', \tilde{Q}$: IN Q'': IN $\hat{Q}$:

$$\dfrac{B, \Pi \longrightarrow \Lambda}{\Pi \longrightarrow \Lambda, \neg B} \quad \cdots\rightarrow \quad \dfrac{B, \Pi, B \longrightarrow \Lambda}{\Pi, B \longrightarrow \Lambda, \neg B} \quad \cdots\rightarrow \quad \dfrac{B, \Pi, B \longrightarrow \Lambda}{\Pi, B \longrightarrow \Lambda} \quad \cdots\rightarrow \quad \dfrac{B, \Pi, B \longrightarrow \Lambda}{\Pi, B \longrightarrow \Lambda}$$

WHAT HAPPENED WITH ENDSEQUENT ?

IN Q: IN $Q', \tilde{Q}$: IN $Q'', \hat{Q}$:

$$\Gamma \longrightarrow \Delta, \neg B \quad \cdots\rightarrow \quad \Gamma, B \longrightarrow \Delta, \neg B \quad \cdots\rightarrow \quad \Gamma, B \longrightarrow \Delta$$

$$\|Q^*\| = \|\hat{Q}\| \leq \|\tilde{Q}\| = \|Q\|$$

SUFFICES $Q^* = \dfrac{\hat{Q}}{B, \Gamma \longrightarrow \Delta}$

CASE $A = \neg B$ CONTINUED:

SIMILARLY WITH SUBPROOF R:

$$R \xrightarrow{\text{ADD } B \text{ TO SUCCEDENT}} R' \xrightarrow{\text{FIX INIT. SEQ.}} \tilde{R} \xrightarrow{\text{REMOVE DIR. ANCESTORS OF } \neg B} R'' \xrightarrow{\text{FIX } \neg\text{:LEFT INFERENCES}} \hat{R}$$

$$R^* = \frac{\hat{R}}{\Gamma \longrightarrow \Delta, B}$$

$$P^* = \frac{Q^* \qquad R^*}{\Gamma \longrightarrow \Delta}$$

$$\|P^*\| = \|Q^*\| + \|R^*\| + 1 \leq \|Q\| + \|P\| + 1 = \|P\|$$

$$= \|P\| < \|P\|^2$$

$\|P\| \geq 3 > 1$

P^* HAS CUTS OF DEPTH $\leq dp(B) = dp(A) - 1$

CASE $A = B \vee C$: P

$$\dfrac{\overbrace{\overset{Q}{\Gamma \longrightarrow \Delta, B \vee C} \qquad \overset{R}{B \vee C, \Gamma \longrightarrow \Delta}}^{P}}{\Gamma \longrightarrow \Delta}\ \text{CUT} \quad \cdots\cdots\!> \quad \dfrac{\dfrac{\overset{Q^*}{\Gamma \longrightarrow \Delta, B, C} \qquad \dfrac{\overset{R_C^*}{C, \Gamma \longrightarrow \Delta}}{C, \Gamma \longrightarrow \Delta, B}}{\Gamma \longrightarrow \Delta, B} \qquad \overset{R_B^*}{B, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta}$$

Q^*: $Q \xrightarrow{\text{ADD } B, C \text{ TO SUCCEDENT}} Q' \xrightarrow{\text{FIX INITIAL SEQUENTS}} \tilde{Q} \xrightarrow{\text{REMOVE DIRECT ANCESTORS OF } B \vee C} Q'' \xrightarrow{\text{FIX } \vee\text{:RIGHT INFERENCES}} \hat{Q} \cdots\cdots> \hat{Q}^*$

VALID — INVALID — VALID — INVALID — VALID — VALID

IN Q: $\dfrac{\Pi \longrightarrow \Lambda, B, C}{\Pi \longrightarrow \Lambda, B \vee C}$ $\cdots\cdots>$ IN $Q', \tilde{Q}$: $\dfrac{\Pi \longrightarrow B, C, \Lambda, B, C}{\Pi \longrightarrow B, C, \Lambda, B \vee C}$ $\cdots\cdots>$ IN Q'': $\dfrac{\Pi \longrightarrow B, C, \Lambda, B, C}{\Pi \longrightarrow B, C, \Lambda}$ $\cdots\cdots>$ IN $\hat{Q}$: $\dfrac{''}{''}$ (double line)

ENDSEQUENT :

$\Gamma \longrightarrow \Delta, B \vee C \cdots\cdots> \Gamma \longrightarrow B, C, \Delta, B \vee C \cdots\cdots> \Gamma \longrightarrow B, C, \Delta$

$$Q = \dfrac{\hat{Q}}{\Pi \longrightarrow \Lambda, B, C}$$

$$\| Q^* \| = \| \hat{Q} \| \leq \| \tilde{Q} \| = \| Q \|$$

CASE $A = B \vee C$ CONTINUED:

R_C^*: $R \xrightarrow[\text{ANTECEDENT}]{\text{ADD } C \text{ TO}} R'_C \xrightarrow{\text{FIX INIT. SEQ.}} \tilde{R}_C \xrightarrow[\text{OF } B\vee C]{\text{REMOVE DIR. ANCESTORS}} R''_C \xrightarrow[\text{INF.}]{\text{FIX } \vee\text{: LEFT}} \hat{R}_C$

IN R:

$$\dfrac{B, \Pi \to \Lambda \qquad C, \Pi \to \Lambda}{B \vee C, \Pi \to \Lambda} \quad \cdots\to$$

IN R'_C, $\tilde{R}_C$

$$\dfrac{B, \Pi, C \to \Lambda \qquad C, \Pi, C \to \Lambda}{B \vee C, \Pi, C \to \Lambda} \quad \cdots\to$$

IN R''_C:

$$\cdots\to \quad \dfrac{B, \Pi, C \to \Lambda \qquad C, \Pi, C \to \Lambda}{\Pi, C \to \Lambda}$$

DISCARD LEFT BRANCH $\cdots\to$ ADD WEAK INFERENCES

IN $\hat{R}_C$

$$\dfrac{C, \Pi, C \to \Lambda}{\Pi, C \to \Lambda} \; \text{WEAK}$$

$$\| R_C^* \| = \| \hat{R}_C \| \le \| \tilde{R}_C \| = \| R \|$$

$$R_C^* = \dfrac{\hat{R}_C}{C, \Pi \to \Lambda}$$

SIMILARLY WITH R_B^*, $\| R_B^* \| \le \| R \|$

$\| Q \|, \| R \| \ge 1$

$$\| P^* \| = \| Q^* \| + \| R_C^* \| + \| R_B^* \| + 2 \le \| Q \| + 2\| R \| + 2 < (\| Q \| + \| R \| + 1)^2 = \| P \|^2$$

P^* HAS CUTS ONLY OF DEPTH $\le dp(B \vee C) - 1$.

CASE $A = (\exists x) B(x)$:

$$\dfrac{\overset{Q}{\Gamma \longrightarrow \Delta, (\exists x) B(x)} \qquad \overset{R}{(\exists x) B(x), \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta} \quad \cdots\cdots> \quad \dfrac{\overset{Q^*}{\vdots}}{\Gamma \longrightarrow \Delta}$$

LET'S ENUMERATE $\exists$:RIGHT INFERENCES IN Q WITH $(\exists x) B(x)$ AS A PRINCIPAL FORMULA:

$$\dfrac{\Pi_i \longrightarrow \Lambda_i, B(t_i)}{\Pi_i \longrightarrow \Lambda_i, (\exists x) B(x)} \qquad i \in \{1, \dots, k\}$$

AND LET'S ENUMERATE $\exists$:LEFT INFERENCES IN R WITH $(\exists x)(B(x))$ AS A P.F.:

$$\dfrac{B(a_i), \Pi_i' \longrightarrow \Lambda_i'}{(\exists x) B(a_i), \Pi_i' \longrightarrow \Lambda_i'} \qquad i \in \{1, \dots, l\}$$

FIRST, TRANSFORM R TO A PROOF R_i OF $B(t_i), \Gamma \longrightarrow \Delta$ FOR EVERY $i \in \{1, \dots, k\}$