Seminar notes (15.11.99), J. Krajíček

$WPHP_n^m(R) := [\exists x < m \forall y < n, \neg R(x,y)] \vee [\exists x < x' < m, y < n, R(x,y) \wedge R(x',y)]$
$R(\log) = \Sigma -$ depth 0 LK $=$ depth 0.5 LK

Theorem(Maciel-Pitassi-Woods): *The propositional version of $WPHP_n^{n^2}(R)$ has quasi-polynomial size $R(\log)$-proofs. In fact, $WPHP_n^{n^2}$ is provable in T_2^2.*

Remarks:(1) This was known for the onto-version of the $WPHP_n^{n^2}$. (2) They prove explicitely the first part; a formalization of the proof in a fragment of bounded arithmetic is just mentioned.

Proof:
For a sequence ω from $\{0,1,\ldots,n\}^*$ define relations $R(\omega,x,y) \subseteq n^2 \times n/(2^\ell)$ by induction on the length ℓ of ω as follows:
$R(\emptyset,x,y) := R(x,y)$.
$R(\omega \cup i, x, y) := \exists z < n, R(x,z) \wedge R(\omega, in+z, y) \wedge y < n/(2^{\ell+1})$, for $0 \le i < n$.
$R(\omega \cup n, x, y) := \exists z < n, R(x,z) \wedge \exists u \in [zn,(z+1)n), R(\omega, u, n/(2^{\ell+1})+y) \wedge y < n/(2^{\ell+1})$.
By induction on ℓ prove:

$$\neg WPHP_n^{n^2}(R) \ \rightarrow \ \exists \omega \in \{0,\ldots,n\}^\ell, \ \neg WPHP_{n/(2^\ell)}^{n^2}(R(\omega,x,y))$$

This is straghtforward as any $S \subseteq n^2 \times n/(2^\ell)$ either (multi-)maps an interval of the form $[in,(i+1)n)$ into $[0,n/(2^{\ell+1}))$ or each such interval contains an element z that is mapped (i.e. has at least one s-value in) to $[n/(2^{\ell+1}), n/(2^\ell))$.

All $R(\omega \cup i, x, y)$ and $R(\omega \cup n, x, y)$ are Σ_1^b in $R(\omega,x,y)$; the witnesses to the Σ_1^b-quantifiers can be pulled together (for $\ell = O(\log n)$) to get a $\Sigma_1^b(R(x,y))$-definition of the ternary relation $R(\omega,x,y)$, for $|\omega| \le \log(n)$.

For a $\Sigma_1^b(R)$-relation the formula $\neg WPHP_n^m$ is $\Pi_2^b(R)$. Hence $\Sigma_3^b(R)$-LIND is enough to carry through the induction. For $\ell = \log(n)$ we get an obvious contradiction, proving thus $WPHP_n^{n^2}(R)$ in $S_2^3(R)$.

We can conclude the proof: The second part of the theorem follows from the $\Sigma_2^b(R)$-conservativity of $S_2^3(R)$ over $T_2^2(R)$ (Buss), the first part follows by the simulation of $T_2^2(R)$ by quasipolynomial $R(\log)$-proofs (Krajíček).

q.e.d.

Remark: The explicit proof manipulations in the original proof correspond to the simulation of the theory by $R(\log)$.