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- In the example above, $A_1, \ldots, A_k$ is called *antecedent* and $B_1, \ldots, B_l$ is called *succedent*. They are both referred to as *cedents*.

Proof system *PK*

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- On next slides, A, B denote formulas and Γ, Δ, etc. denote cedents.

Weak structural rules

$$\text{Exchange:left}\ \frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta} \qquad \text{Exchange:right}\ \frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda}$$

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$$\text{Contraction:left}\ \frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \qquad \text{Contraction:right}\ \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}$$

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All other inference rules are called *strong*.

Cut rule

$$\text{cut}\ \frac{\Gamma \to \Delta, A \qquad A, \Gamma \to \Delta}{\Gamma \to \Delta}$$

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A proof in *PK* is *cut free*, if it does not use the cut rule.

Propositional rules

$$\neg\text{:left}\ \frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta} \qquad \neg\text{:right}\ \frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$$

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$$\supset\text{:left}\ \frac{\Gamma \rightarrow \Delta, A \qquad B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta} \qquad \supset\text{:right}\ \frac{A, \Gamma\rightarrow\Delta, B}{\Gamma\rightarrow\Delta, A \supset B}$$

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- C is an (direct, immediate) *ancestor* of D, if D is a (direct, immediate) descendant of C.

Example proof

$$a \vee b, \neg a \vee c \rightarrow b \vee c$$

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$$\vee\text{:right}\ \frac{a \vee b, \neg a \vee c \rightarrow b, c}{a \vee b, \neg a \vee c \rightarrow b \vee c}$$

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$$\vee\text{:right}\;\dfrac{\vee\text{:left}\;\dfrac{a, \neg a \vee c \rightarrow b, c \qquad b, \neg a \vee c \rightarrow b, c}{a \vee b, \neg a \vee c \rightarrow b, c}}{a \vee b, \neg a \vee c \rightarrow b \vee c}$$

Example proof

$$
\vee\text{:right}\ \cfrac{\vee\text{:left}\ \cfrac{a, \neg a \vee c \to b, c \qquad \cfrac{b \to b}{b, \neg a \vee c \to b, c}}{a \vee b, \neg a \vee c \to b, c}}{a \vee b, \neg a \vee c \to b \vee c}
$$

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      \qquad
      \dfrac{c \to c}{a, c \to b, c}
    }{a, \neg a \vee c \to b, c}
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Theorem (subformula property)

If P is a cut free PK-proof, then every formula occuring in P is a subformula of a formula in the endsequent of P.

Proof

Since the proof is cut free, all formulas in all sequents except the endsequent have an immediate descendant. Thus, every formula has a descendant in the endsequent.

Soundness and completeness

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Completeness theorem

Every valid sequent has a cut free proof in PK.

Proof length

We distinguish between ‘tree-like’ and ‘dag-like’ proofs (‘dag’ stands for ‘directed acyclic graph’). Unless stated otherwise, all proofs are presumed to be tree-like.

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Theorem

For a given tree-like proof P of sequent $\Gamma \to \Delta$, there is a tree-like proof of $\Gamma' \to \Delta'$ for some $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ having at most $\|P\|^2$ sequents.

Completeness theorem

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Let $\Gamma \to \Delta$ be a valid sequent with m occurences of logical connectives. Then there is a tree-like cut free PK-proof P of $\Gamma \to \Delta$ such that $\|P\| < 2^m$.

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The proof is by induction on m. For $m = 0$ all formulas in Γ, Δ are atomic. Since $\Gamma \to \Delta$ is valid, there is some variable p which occurs both in Γ and Δ. Thus $\Gamma \to \Delta$ can be proved with zero strong inferences from the initial sequent $p \to p$.

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Now let $m \geq 1$. Assume the sequent is of the form $\neg A, \Gamma' \to \Delta$ for some formula A. Then $\Gamma' \to \Delta, A$ is valid, and by induction hypothesis it can be proved in less than 2^{m-1} strong inferences. Using $\neg$:left, we can thus prove our sequent in less than $2^{m-1} + 1 \leq 2^m$ strong inferences.

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Other cases are handled analogously by using $\vee$:left, $\vee$:right, $\supset$:left and $\supset$:right. The inversion theorem implies that we never attempt to prove a sequent which is not valid.

Cut rule

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Cut-elimination theorem

Let P be a dag-like proof of $\Gamma \rightarrow \Delta$. Then there is a tree-like cut free proof Q of $\Gamma \rightarrow \Delta$ such that $\|Q\| \leq 2^{\|P\|_{dag}}$.

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Free cuts

Let P be a $\mathfrak{S}$-proof and let I be a cut inference in P. We say that I's cut formulas are *directly descended from* $\mathfrak{S}$, if they have some direct ancestor in an initial sequent from $\mathfrak{S}$. A cut I is *free* if neither of I's cut formulas are directly descended from $\mathfrak{S}$. A proof is *free-cut free*, if it contains no free cuts.

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Free-cut elimination theorem

Let S be a sequent and $\mathfrak{S}$ a set of sequents. If $\mathfrak{S} \models S$, then there is a free-cut free $\mathfrak{S}$-proof of S.

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- Cut elimination theorem for Tait calculus is called *normalization theorem*. For general infinitary logic it does not hold. However, it holds for logic with formulas of countable length.

Thank you for your attention