Cut-Elimination for LK-Calculus

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Theorem

Let P be an LK-proof and suppose every cut formula in P has depth less than or equal to d. Then there is a cut-free LK-proof P^* with the same endsequent as P, with size

$$\|P^*\| < 2^{\|P\|}_{2d+2}.$$

Lemma

Let P be an LK-proof with final inference a cut of depth d such that every other cut in P has depth strictly less than d. Then there is an LK-proof P^* with the same endsequent as P with all cuts in P^* of depth less than d and with $\|P^*\| < \|P\|^2$.

Proof.

The proof P ends with a cut inference

$$\dfrac{\dfrac{\vdots\, Q}{\Gamma \longrightarrow \Delta, A} \qquad \dfrac{\vdots\, R}{A, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta}$$

where the depth of the cut formula equals d and where all cuts in the subproofs Q and R have depth strictly less than d. The proof of this theorem is by induction on the outermost logical connective of the cut formula A. □

Proof (Cont.)

The proof for the cases of $A = \neg B, B \vee C, B \wedge C$, and $B \supset C$ are done in previous two lectures. We still have the cases where A are of the form $(\exists x)B(x)$ and $(\forall x)B(x)$. We prove the case where A is $(\exists x)B(x)$ since the proof of the case $(\forall x)B(x)$ is similar.

Subproof Q

Since A is not atomic, it can only be introduced by weakening and by $\exists$:*right* inferences. Suppose that there are $k \geq 0$ many $\exists$:*right* inferences in Q which have their principal formula a direct ancestor of the cut formula. List as

$$\frac{\Pi_i \to \Lambda_i, B(t_i)}{\Pi_i \to \Lambda_i, (\exists x)B(x)}$$

for $1 \leq i \leq k$.

Proof (Cont.)

Subproof R

Suppose that there are $l \geq 0$ many $\exists$:*left* inferences in R which have their principal formula a direct ancestor of the cut formula. List as

$$\frac{B(a_i), \Pi'_i \to \Lambda'_i}{(\exists x)B(x), \Pi'_i \to \Lambda'_i}$$

for $1 \leq i \leq l$.

Idea : Construct new proof based on the proof we already have.

Proof (Cont.)

For each $i \leq k$, we form a proof R_i of the sequent $B(t_i), \Gamma \to \Delta$ as follows :

- In R, replacing all l variables a_i with the term t_i,
- In R, replacing every direct ancestor of the cut formula $(\exists x)B(x)$ with $B(t_i)$,
- Removing the l-many $\exists$:*left* inferences.

Remark

P is in free variable normal form ensures that replacing the a_i's with t_i will not impact the eigenvariable condition.

Proof (Cont.)

Construct Q' from subproof Q as follows :

- Replacing each sequent $\Pi \to \Lambda$ in Q with the sequent $\Pi, \Gamma \to \Delta, \Lambda^-$ where $\Lambda^- := \Lambda$ minus all direct ancestors of $(\exists x)B(x)$. Ex. the end sequent is $\Gamma, \Gamma \to \Delta, \Delta$
- Initial Sequent : $A, \Gamma \to \Delta, A$. Can be derived by $A \to A$ using weakenings and exchanges.
- For each $1 \le i \le k$, replace i-th $\exists$:*right* inference :

$$\frac{\Pi_i, \Gamma \to \Delta, \Lambda_i, B(t_i)}{\Pi_i, \Gamma \to \Delta, \Lambda_i}$$

by

$$\frac{\Pi_i, \Gamma \longrightarrow \Delta, \Lambda_i, B(t_i) \qquad \overset{\vdots\, R_i}{B(t_i), \Gamma \longrightarrow \Delta}}{\Pi_i, \Gamma \longrightarrow \Delta, \Lambda_i}$$

Proof (Cont.)

- Construct P^* from Q' by adding some exchanges and contractions to the end of Q'. This gives us new proof P^* of $\Gamma \to \Delta$.
- Note that the replacement of $\exists$:*right* inference of Q above gives us cut inference with a cut of depth $d-1$,
- Every cut in P^* has depth $< d$,
- $\|P^*\| \leq \|Q\| \cdot (\|R\| + 1) < \|P\|^2$.

From the previous lemma, we can replace a single cut by lower depth cut inferences. Iterating this construction, we can remove all cuts of the maximum depth d in a proof.

Lemma

If P is an LK-proof with all cuts of depth at most d, there is an LK-proof with the same endsequent which has all cuts of depth strictly less than d and with size $\|P^*\| < 2^{2^{\|P\|}}$.

Proof.

This can be proved by induction on the number of depth d cuts in P.

- Base case : No depth d cuts. We get P^* which is P and $\|P\| < 2^{2^{\|P\|}}$.
- Inductive case : it suffices to prove the lemma in the case where P ends with the following sequent with cut formula A of the depth d

$$\dfrac{\overset{\vdots\, Q}{\Gamma \longrightarrow \Delta, A} \qquad \overset{\vdots\, R}{A, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta}$$

□

Proof (Cont.)

Subproof R with $\|R\| = 0$

R must satisfies one of the following cases :

1. containing the axiom $A \to A$
2. having direct ancestors of the cut formula A introduced by weakenings

Then

1. the cut formula A must appear in Δ, and the proof P^* can be obtained from Q by adding some exchange inferences and a contraction inference to the end of Q,
2. the proof P^* can be obtained from R by removing all the *Weakening:left* inferences that introduce direct ancestors of the cut formula A,

Subproof Q with $\|Q\| = 0$: Similar.

Note. $\|P^*\| < \|P\| < 2^{2^{\|P\|}}$

Proof (Cont.)

Subproof R and Q with $\|R\| \neq 0, \|Q\| \neq 0$

By inductive hypothesis, there are proof Q^* and R^* of the same sequents, with all cuts of depth $< d$, and

$$\|Q^*\| < 2^{2^{\|Q\|}}, \|R^*\| < 2^{2^{\|R\|}}.$$

Applying previous lemma to the proof

$$\dfrac{\overset{\vdots\; Q^*}{\Gamma \longrightarrow \Delta, A} \qquad \overset{\vdots\; R^*}{A, \Gamma \longrightarrow \Delta}}{\Gamma \longrightarrow \Delta}$$

gives a proof P^* of $\Gamma \rightarrow \Delta$ with all cuts of depth $< d$. Note that $\|P^*\| < (\|Q^*\| + \|R^*\| + 1)^2 \leq (2^{2^{\|Q\|}} + 2^{2^{\|R\|}} - 1)^2 < 2^{2^{\|Q\|+\|R\|+1}} = 2^{2^{\|P\|}}$.

A general bound on cut elimination

The upper bound $2^{\|P\|}_{2d+2}$ in the Cut Elimination Theorem is based not only on the size of P , but also on the maximum depth of the cut formulas in P.

Proposition

Suppose P is an LK-proof of the sequent $\Gamma \to \Delta$. Then there is a cut-free proof P^* of the same sequent with size $\|P^*\| < 2^{\|P\|}_{2\|P\|}$.