Hintikka Games and Game-Theoretical Semantics

Maroš Grego

<2021-05-18 Tue>

Motivation: the limit definition

The number A is a limit of a real function $f(x)$ at x_0 if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)(|x - x_0| < \delta \rightarrow |f(x) - A| < \epsilon)$$

- can be understood as a game of 2 players trying to get arbitrarily close to A

Let L be a first-order language, M a model of L, S a sentence of L. A semantical game $G_M(S)$ of players *Verifier*, *Falsifier* is played by these rules:

$(R.\vee)$	$G_M((S_1 \vee S_2))$ - Verifier picks $i = 1, 2$ continues as $G(S_i)$
$(R.\wedge)$	$G_M((S_1 \wedge S_2))$ - Falsifier picks $i = 1, 2$ continues as $G(S_i)$
$(R.\exists)$	$G_M((\exists x)(S_0[x]))$ - Verfier picks b in $dom(M)$ continues as $G(S_0[b])$
$(R.\forall)$	$G_M((\forall x)(S_0[x]))$ - Falsifier picks b in the $dom(M)$ continues as $G(S_0[b])$
$(R.\neg)$	$G_M(\neg S_0)$ is like $G(S_0)$ with player roles reversed
$(R.atom)$	S atomic - Verifier wins if S is true, Falsifier if false

Definition (Truth in GTS)

A sentence S is true in a model M ($M \models_{GTS} S^{+}$) if there exists a winnig strategy for Verifier in $G_M(S)$.
A sentence S is false in a model M ($M \models_{GTS} S^{-}$) if there exists a winnig strategy for Falsifier in $G_M(S)$.

Theorem (GTS and Tarski equivalence)

Assuming Axiom of Choice, for every first-order sentence S and model M, the Tarski and GTS definitions of truth coincide ($M \models_{Tarski} S$ iff $M \models_{GTS} S$).

Proof.

Inductively by the sentence size. AC is needed to choose the strategy. □

Theorem (Skolem functions)

Every first order sentence S is equivalent to a second order Σ^1_1 existential sentence.

Proof.

- transform S into its negation normal form S_n
- replace each variable x bound by $\exists$ in S_n by $F(y_1, y_2, ...)$, where F is a new function symbol and $(\forall y_1)$, $(\forall y_2)$, ... are universal quantifiers in scope of which x occurs
- replace each $(S_1 \vee S_2)$ by $(G(y_1, y_2, ...) = 0 \wedge S_1) \vee (G(y_1, y_2, ...) \neq 0 \wedge S_2)$, where G is a new function symbol and y_1, y_2, ... as above
- bound the newly introduced function variables to initial quantifiers

$\square$

Example (Simple relation)

$(\forall x)(\exists y)(\forall z)(\exists w)(R[x, y, z, w])$ is transformed into $(\exists F_1)(\exists F_2)(\forall x)(\forall z)(R[x, F_1(x), z, F_2(x, z)])$

- What about Σ^1_1 formulas of this form, whose funcion symbols do not depend on all quantifiers in the sequence, such as $(\exists F_1)(\exists F_2)(\forall x)(\forall z)(R[x, F_1(x), z, F_2(x, z)])$?
- These can't be in general equivalent to ordinary first order formulas, since there, the scope of 2 quantifiers is either disjoint or nested:

$$\underline{(\forall x)(\exists y)\underline{(\forall z)(\exists w)(R[x, y, z, w])}}$$

What about scopes like

$$\underline{(\forall x)(\exists y)}\underline{(\forall z)(\exists w)}(R[\underline{x, y}, \underline{z, w}])$$

Independence Friendly (IF) first-order logic

Ordinary first order logic extended with / symbol.

- $(Q_1 x / Q_2 y)$ means the variable x under the quantifier Q_1 is independent of the variable y under the quantifier Q_2
- In GTS, that means the player picking x can't use y for their strategy (the game is not of perfect information)

Example (Simple formula)

$$(\forall x)(\forall z)(\exists y/\forall z)(\exists w/\forall x)(R[x, y, z, w])$$

IF first-order logic

Example (Alternative notation)

$$\begin{matrix} \forall x & \exists y \\ \forall z & \exists w \end{matrix} R[x, y, z, w]$$

IF first-order logic

- Independence can be extended to cover all logical constants.
- The usual first-order logic formation rules are extended with these

IF formation rules

If $(\Box)$ occurs with the scope of $(Q_1y_1), (Q_2y_2), \ldots$ in a first-order formula, where $\Box$ can be one of $\forall x, \exists x, \wedge, \vee$, it can be replaced by $(\Box / Q_1y_1, Q_2y_2, ...)$

Theorem (Hintikka, Sandu)

Every IF first-order sentence is equivalent with a Σ^1_1 sentence.

Proof.

Use strategy functions as in ordinary first-order logic. □

Theorem (Enderton, Hintikka)

Every Σ^1_1 sentence S is equivalent to an IF first-order sentence.

Proof.

- By Skolem functions and quantifier tricks, bring S to the form $\exists F_1 \exists F_2 ... \forall x_1 \forall x_2 ... S'$ where S' is quantifier-free
- Eliminate nested function symbols by replacing e.g. $\phi[F_i(t)]$ with $\forall u(u = t \rightarrow \phi[F_i(u)])$
- Ensure every function symbol occurs with the same variables, e.g. by replacing $\exists F \forall x \forall y \phi[F(x), F(y)]$ with $\exists F \exists G \forall x \forall y (x = y \rightarrow F(x) = G(y)) \wedge \phi[F(x), G(y)]$
- Sentences of this form can be straightforwardly translated into IF first-order logic

□

Theorem (IF first-order logic properties)

IF first-order logic is not recursively axiomizable, but compact extension of ordinary first-order logic.

Proof.

With the equivalence of IF first-order logic and Σ_1^1 logic, we get for the former the meta-logical properties of the later. □

Separation Theorem; Barwise

Theorem (Barwise)

For K_1 and K_2 disjoint classes of structures definable by IF first-order language, there is an elementary class K (definable by a single ordinary first-order sentence) such that K contains K_1 but is disjoint from K_2.

The failure of law of the excluded middle

- Consider the semantical game on the sentence $(\forall x)(\exists y/\forall x)(x = y)$
- It has no winning strategy for either player on any domain with more than one element

Definition (Weak negation)

Extend an IF first-language with a logical constant $\neg_w$, which can only occur at the start of a sentence.
Given a sentencte S and a model M,
$M \models_{GTS} (\neg_w S)^+$ if not $M \models_{GTS} S^+$ (Verifier has no winning strategy)
$M \models_{GTS} (\neg_w S)^-$ if not $M \models_{GTS} S^-$ (Falsifier has no winning strategy)

Theorem (Hintikka)

For any sentence S of an IF first-order language L, if $\neg_w S$ is representable in L (i.e. there is an L-sentence R such that S and R have the same models), then S is representable by an ordinary first order sentence.

Proof.

Follows from the Separation Theorem. □

Definability of truth

Let L be an ordinary first-order arithmetical language and let $\ulcorner S \urcorner$ denote the Gödel number of S and $\bar{n}$ the numeral of n. Let a *truth predicate* be a second order predicate $(\exists X)(Tr[X] \wedge X(y))$, where $Tr[X]$ is a conjunction of

- $\forall x \forall y \forall z((x = \ulcorner (S_1 \wedge S_2) \urcorner \wedge y = \ulcorner S_1 \urcorner \wedge z = \ulcorner S_2 \urcorner) \rightarrow (X(x) \rightarrow X(y) \wedge X(z)))$, analog. for disjunction
- $\forall y \forall z \forall w((x = \ulcorner \forall x S[x] \urcorner \wedge w = \ulcorner S[\bar{z}] \urcorner \wedge X(y)) \rightarrow X(w))$, analog. for existential quantifier
- $\forall x \forall y(X(\ulcorner R(\bar{x}, \bar{y}) \urcorner) \leftrightarrow R(x, y))$ or similar for primitive and negated primitive predicates
- $\forall x \forall y(N(x, y) \rightarrow (X(x) \leftrightarrow X(y)))$, where N is a relation of Gödel numbers of a sentence and their negation normal form

Definability of truth

- Property of being true satisfies $Tr[X]$; conversely, if the truth predicate is true of $\ulcorner S \urcorner$, it defines a winning stratery for Verifier
- The truth predicate is a Σ^1_1 formula, so it can be translated into the IF extension of L.
- The truth predicate can be extended to a language L where arithmetic can be represented by defining it as $(\exists F)(Sat(y, F))$, where F is a valuation function and Sat is a satisfaction relation.

Definability of truth for IF languages

Let L be an IF first-order arithmetical language.

- Express that X applies to the Gödel number of a sentence iff it applies to its Skolem normal form
- Express that X applies to a sentence it Skolem normal form

  $$(\forall x_1)(\forall x_2)...(\exists y_1/\forall x_{11}\forall x_1 2...)...R[x_1, x_2, ..., y_1, ...]$$

  only if there are functions $F_1, F_2, \ldots$ such that X applies to the Gödel number of every sentencte of a form $R[\bar{n}_1, \bar{n}_2, ..., \overline{f_1(n_{11}, n_{12}, ...)}, ...]$.

Definability of truth for IF languages

- All of those requirements are Σ_1^1 formulas. Denote their conjunction $Tr[X]$ and consider $(\exists X)(Tr[X] \wedge X(y))$
- This predicate is Σ_1^1 and can be translated into IF first-order language
- Can be generalised to more languages similar to the ordinary first-order case

Thank you!