Sequent calculus for bounded arithmetic and intro to witnessing theorems

Jiří Rýdl

April 28, 2022

Outline

1. Sequent calculus for arithmetic theories
2. Witnessing theorem for $I\Sigma_1$

Sequent calculus for arithmetic theories

To obtain a sequent calculus formulation of arithmetic theories, the calculus LK is extended by rules for IND, MIN, REP and rules for bounded quantifiers:

$$\mathrm{L}\forall\leq \qquad \frac{A(t), \Gamma \Rightarrow \Delta}{t \leq s, (\forall x \leq s)A(x), \Gamma \Rightarrow \Delta}$$

$$\mathrm{R}\forall\leq \qquad \frac{b \leq s, \Gamma \Rightarrow \Delta, A(b)}{\Gamma \Rightarrow \Delta, (\forall x \leq s)A(x)}$$

Sequent calculus for arithmetic theories

$$\text{L}\exists\leq \quad \frac{b \leq s, A(b), \Gamma \Rightarrow \Delta}{(\exists x \leq s)A(x), \Gamma \Rightarrow \Delta}$$

$$\text{R}\exists\leq \quad \frac{\Gamma \Rightarrow \Delta, A(t)}{t \leq s, \Gamma \Rightarrow \Delta, (\exists x \leq s)A(x)}$$

The variable b works as an eigenvariable (it does not occur in the contexts). LK plus the above four rules is called LKB and the Free-cut Elimination Theorem holds for it, principal formulas of the rules are $t \leq s$ and $(Qx \leq s)A$.

Induction rules

The reason for taking induction rules instead of induction axioms is that the Free-cut Elimination Theorem will still hold. With the contexts Γ, Δ, the rules turn out to be equivalent to the axioms.

Φ-IND induction

$$\frac{A(b), \Gamma \Rightarrow \Delta, A(b+1)}{A(0), \Gamma \Rightarrow \Delta, A(t)}$$

Φ-PIND induction

$$\frac{A(\lfloor \frac{1}{2} b \rfloor), \Gamma \Rightarrow \Delta, A(b)}{A(0), \Gamma \Rightarrow \Delta, A(t)}$$

In both cases, b works as an eigenvariable.

Subformula property for fragments of PA

Theorem

Let Φ be a class of formulas containing atomic formulas and being closed under subformulas and term substitution. Let R be an arithmetic theory axiomatized by Φ-IND (or Φ-PIND) rules plus initial sequents containing formulas from Φ. Suppose that $\Gamma \Rightarrow \Delta$ contains only formulas from Φ and that R proves $\Gamma \Rightarrow \Delta$. Then there is an R-proof of $\Gamma \Rightarrow \Delta$ such that every formula in that proof is in Φ.

Parikh's theorem

- let a bounded theory R contain $\leq$ in its language and let the reflexivity and transitivity of $\leq$ be provable in R
- assume further that for all terms r, s there is a term t such that $R \vdash r \leq t$ and $R \vdash s \leq t$
- lastly assume that for all terms $t(b)$ and r (possibly with parameters) there is a term s such that $R \vdash b \leq r \to t(b) \leq s$

Theorem (Parikh)

If R is a bounded theory satisfying the above conditions, $A(\vec{x}, y)$ is a bounded formula and $R \vdash (\forall \vec{x})(\exists y)A(\vec{x}, y)$, then there is a term t such that $R \vdash (\forall \vec{x})(\exists y \leq t)A(\vec{x}, y)$.

Proof outline

- work with a free-cut free R-proof P of the formula $(\exists y)A(\vec{b}, y)$, where all b's are new variables
- by the subformula property, all antecedents in P contain only bounded formulas, and the succedents can, beside that, only contain occurrences of the formula $(\exists y)A(\vec{b}, y)$
- prove by induction on the number of inferences in P that for every sequent $\Gamma \Rightarrow \Delta$ in P there is a term t such that R proves $\Gamma \Rightarrow \Delta_t$, where Δ_t denotes Δ without all occurrences of $(\exists y)A(\vec{b}, y)$ and with one occurrence of $(\exists y \leq t)A(\vec{b}, y)$

Inference rules for collection

The Σ_1 -collection rules (Σ_1-REPL) are the following inferences:

$$\frac{\Gamma \Rightarrow \Delta, (\forall x \leq t)(\exists y) A(x, y)}{\Gamma \Rightarrow \Delta, (\exists z)(\forall x \leq t)(\exists y \leq z) A(x, y)}$$

It holds that:

- the REPL rules and axioms are equivalent
- the Free-cut Elimination Theorem holds (provided every direct descendant of the principal formula of every REPL inference is taken to be anchored)
- the corollary is that Parikh's theorem also holds for theories containing Σ_1-REPL

Witnessing theorem for $I\Sigma_1$

We want to prove the following theorem:

Theorem (Parsons (1970), Mints(1973) and Takeuti (1987))

Every Σ_1-definable function of $I\Sigma_1$ is primitive recursive.

The method Buss uses is the *witnessing theorem method*, and it is claimed that "$I\Sigma_1$ provides the simplest and most natural application of the witnessing method".

Proof outline

- suppose $I\Sigma_1$ proves a formula $(\forall x)(\exists y)A(x,y)$ for some $A \in \Sigma_1$
- then there is a sequent calculus proof of $(\exists y)A(c,y)$
- we must prove that there is a p.r. function f such that for every n: $A(n, f(n))$ holds
- a corollary of the following lemma gives something stronger, namely that there is a Σ_1-definable p. r. function f such that $I\Sigma_1$ proves $(\forall x)A(x, f(x))$

The witness predicate for Σ_1-formulas

If $A(\vec{b})$ is a Σ_1-formula of the form $(\exists x_1, \ldots, x_k)B(x_1, \ldots, x_k, \vec{b})$ with $B \in \Delta_0$, define $Witness_A(w, \vec{b})$ to be the formula

$$B(\beta(1, w), \ldots, \beta(k, w), \vec{b})$$

If $\Delta = \Delta', A$ is a succedent, then $Witness_{\bigvee \Delta}(w, \vec{c})$ is the formula

$$Witness_A(\beta(1, w), \vec{c}) \vee Witness_{\bigvee \Delta'}(\beta(2, w), \vec{c})$$

If $\Gamma = \Gamma', A$ is an antecedent, then $Witness_{\bigwedge \Gamma}(w, \vec{c})$ is the formula

$$Witness_A(\beta(1, w), \vec{c}) \wedge Witness_{\bigwedge \Gamma'}(\beta(2, w), \vec{c})$$

Intuitively, $Witness_A(w, \vec{b})$ is a formula stating that w is a witness for the truth of A. It is a Δ_0-formula and $I\Delta_0$ can prove

$$A(\vec{b}) \leftrightarrow (\exists w) Witness_A(w, \vec{b})$$

Witnessing lemma for $I\Sigma_1$

Lemma

Suppose $I\Sigma_1$ proves a sequent $\Gamma \Rightarrow \Delta$ of Σ_1-formulas. Then there is a function h such that:

1. *h is Σ_1-defined by $I\Sigma_1$ and p. r.*
2. *$I\Sigma_1$ proves*

$$(\forall \vec{c})(\forall w)[Witness_{\bigwedge \Gamma}(w, \vec{c}) \Rightarrow Witness_{\bigvee \Delta}(h(w, \vec{c}), \vec{c})]$$

Corollary

If we let Γ be empty and Δ consist only of the formula $(\exists y)A(c, y)$ and set $f(x) = \beta(1, \beta(1, h(x)))$, the above theorem follows.

Proof of the lemma

We work with a free-cut free proof P (in $I\Sigma_1$) of the sequent $\Gamma \Rightarrow \Delta$, where whose every formula we can assume to be Σ_1. The proof is by induction of the number of inferences in P.

First let the last inference be $R\exists$ on the formula A:

$$\frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, (\exists x)A(x)}$$

Proof of the lemma - case R∃

The induction hypothesis gives a Σ_1-defined p. r. function $g(w, \vec{c})$ such that $I\Sigma_1$ proves

$$Witness_{\bigwedge \Gamma}(w, \vec{c}) \Rightarrow Witness_{\bigvee\{\Delta, A(t)\}}(g(w, \vec{c}), \vec{c})$$

For the succedent to hold we must have that either $\beta(2, g(w, \vec{c}))$ witnesses $\bigvee \Delta$ or that $\beta(1, g(w, \vec{c}))$ witnesses $A(t)$.
Define

$$h(w, \vec{c}) = \langle\langle t(\vec{c})\rangle * \beta(1, g(w, \vec{c}), \beta(2, g(w, \vec{c}))\rangle$$

From the definition of *Witness* is follows that

$$Witness_{\bigwedge \Gamma}(w, \vec{c}) \Rightarrow Witness_{\bigvee\{\Delta, (\exists x)A(x)\}}(h(w, \vec{c}), \vec{c})$$

Proof of the lemma - case L∃

Suppose the last inference is L∃ on $A(b)$, where b is an eigenvariable. The induction hypothesis gives a Σ_1-defined p. r. function $g(w, \vec{c}, b)$ such that $I\Sigma_1$ proves

$$Witness_{\bigwedge\{A(b),\Gamma\}}(w, \vec{c}) \Rightarrow Witness_{\bigvee\Delta}(g(w, \vec{c}, b), \vec{c})$$

Denote the function $tail(\langle w_0, w_1, \ldots, w_n\rangle) = \langle w_1, \ldots, w_n\rangle$ by $tail(w)$ and let $h(w, \vec{c})$ be the function

$$g(\langle tail(\beta(1, w)), \beta(2, w)\rangle, \vec{c}, \beta(1, \beta(1, w)))$$

Then h satisfies the conditions of the lemma.

Proof of the lemma - case IND

Suppose the final inference is a Σ_1-IND inference step:

$$\frac{A(b), \Gamma \Rightarrow \Delta, A(b+1)}{A(0), \Gamma \Rightarrow \Delta, A(t)}$$

The induction hypothesis gives a Σ_1-defined p. r. function $g(w, \vec{c}, b)$ such that $I\Sigma_1$ proves

$$Witness_{\bigwedge\{A(b),\Gamma\}}(w, \vec{c}, b) \Rightarrow Witness_{\bigvee\{\Delta, A(b+1)\}}(g(w, \vec{c}, b), \vec{c}, b)$$

Proof of the lemma - case IND (cont.)

Define the following auxiliary function:

$$k(\vec{c}, v, w) = \begin{cases} v, & \text{if } Witness_\Delta(v, \vec{c}) \\ w, & \text{otherwise} \end{cases} \tag{1}$$

This function is Σ_1-defined by $I\Sigma_1$, because $Witness \in \Delta_0$.
Now define $f(w, \vec{c}, b)$:

$$f(w, \vec{c}, 0) = \langle \beta(1, w), 0 \rangle$$

$$f(w, \vec{c}, b+1) =$$

$$\langle \beta(1, g(\langle \beta(1, f(w, \vec{c}, b)), \beta(2, w) \rangle, \vec{c}, b)),$$

$$k(\vec{c}, \beta(2, f(w, \vec{c}, b)), \beta(2, g(\langle \beta(1, f(w, \vec{c}, b)), \beta(2, w) \rangle, \vec{c}, b))) \rangle$$

Proof of the lemma - case IND (cont.)

Since f is p. r., it is Σ_1-definable by $I\Sigma_1$ and can be used in induction formulas, we can use Σ_1-IND w. r. t. b to conclude

$$Witness_{\bigwedge\{A(0),\Gamma\}}(w,\vec{c}) \Rightarrow Witness_{\bigvee\{\Delta,A(b)\}}(f(w,\vec{c},b),\vec{c},b)$$

Now we can set $h(w,\vec{c}) = f(w,\vec{c},t)$ and this function satisfies the conditions of the lemma.

Corollary

Theorem

The Δ_1-defined predicates of $I\Sigma_1$ are precisely the p. r. predicates.

Proof.

Suppose $A(c)$ and $B(c)$ are Σ_1-formulas such that $I\Sigma_1$ proves

$$(\forall x)(A(x) \leftrightarrow \neg B(x))$$

Then $Char_A$ is Σ_1-definable in $I\Sigma_1$, because $I\Sigma_1$ proves

$$(\forall x)(\exists! y)[(A(x) \wedge y = 0) \vee (B(x) \wedge y = 1)]$$

By the above theorem $Char_A$ is primitive recursive, and hence so is the predicate $A(c)$. □