Overview of bootstrapping (phase 2) and relationships among stronger fragments

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Outline

A theorem on Σ_1-defined functions

Bootstrapping $I\Delta_0$, phase 2 (coding sequences) - brief overview

Relationships amongst the axioms of PA

Review

$$A(0) \wedge (\forall x)(A(x) \rightarrow A(x+1)) \rightarrow (\forall x)A(x) \qquad \text{(IND)}$$

$$(\exists x)A(x) \rightarrow (\exists x)(A(x) \wedge \neg(\exists y)(y < x) \wedge A(y)) \qquad \text{(LNP)}$$

$$(\forall x \leq t)(\exists y)A(x,y) \rightarrow (\exists z)(\forall x \leq t)(\exists y \leq z)A(x,y) \qquad \text{(REPL)}$$

Definition

$B\Sigma_n$ is the theory $I\Delta_0$ plus all Σ_n-REPL axioms, i.e. all instances of REPL for $A \in \Sigma_n$, and similarly for $B\Pi_n$

Review

Definition

A predicate $R(\vec{x})$ is Δ_0-*defined* if there is a formula $\varphi(\vec{x}) \in \Delta_0$ and a defining axiom $R(\vec{x}) \leftrightarrow \varphi(\vec{x})$.

A function symbol $f(\vec{x})$ is Σ_1-*defined* by a theory of arithmetic T if $y = f(\vec{x}) \leftrightarrow \varphi(\vec{x}, y)$ for a Σ_1 formula φ is its defining axiom and

$$T \vdash (\forall \vec{x})(\exists! y)\varphi(\vec{x}, y)$$

Theorem

$f(\vec{x})$ is Σ_1-defined by $I\Delta_0 \Leftrightarrow$ its defining formula $\varphi(\vec{x})$ is Δ_0 and there is a bounding term $t(\vec{x})$ such that

$$I\Delta_0 \vdash (\forall \vec{x})(\exists! y \leq t)\varphi(\vec{x}, y)$$

A theorem on Σ_1-definable functions

Theorem

If T^+ is a theory extending some bounded theory $T \supseteq Q$ by adding Δ_0-defined predicates and Σ_1-defined function symbols and their defining equations, then T^+ is conservative over T. Also, if A is a formula possibly containing some of the new function or predicate symbols, then there is A^- in the language of T such that

$$T^+ \vdash A \leftrightarrow A^-$$

This also holds for $T \supseteq B\Sigma_1$ and Δ_1-defined predicates with the addition that if A is Σ_n (Π_n), then A^- is also Σ_n (Π_n), respectively.

Proof - first part

We show that the new function and predicate symbols can be eliminated from A without increase in the (unbounded) quantifier complexity in such a way that the T^+-equivalence is preserved.

- Δ_0-defined predicates can be replaced by their defining formulas
- eliminate new function symbols from bounded quantifiers by replacing each $(\forall x \leq t)(\dots)$ by $(\forall x \leq t^*)(x \leq t \to \dots)$, where t^* is obtained from t by replacing every new function symbol with its bounding term
- and do the same operation with the bounded existential quantifiers that contain some of the new function symbols

Proof - first part

- if f is a new function symbol, replace every atomic formula $P(f(y))$ by one of the following two formulas:

$$(\exists z \leq t(y))(A_f(y,z) \wedge P(z))$$

$$(\forall z \leq t(y))(A_f(y,z) \rightarrow P(z))$$

  where A_f is a formula which defines f and t is a bounding term of f
- because $T \vdash (\forall x)(\exists! y)A_f(x,y)$, the formulas above are equivalent to $P(f(y))$ in T^+

Proof - notes on the second part

There are some modifications:

- as the theories are stronger than $I\Delta_0$, there is no bounding term t, so the two formulas replacing an atomic formula use an unbounded quantification, and are thus in Σ_n or Π_n
- but since A is in Σ_n or Π_n, there is always a choice that does not increase the number of alternating unbounded quantifiers
- the second thing is that we need Σ_1-replacement axioms for the elimination of the new function symbols from terms in bounded quantification

Corollary of the previous theorem

Theorem

Let T be $I\Delta_0, I\Sigma_n$ or $B\Sigma_n$, then in the conservative extension T^+ we may use the new function and relation symbols freely in induction, minimization and replacement axioms.

The aim of bootstrapping, phase 2

- we want to formalize sequences inside $I\Delta_0$, i.e. we want code sequences of numbers as numerals and have formulas expressing concepts such as "the *i*-th entry of the sequence coded by x is y" (Gödel's beta function)
- also we need to be able to prove in $I\Delta_0$ that the respective notions have properties which we would expect
- the central difficulty is that one has to carefully choose *how* the relevant concepts are defined, because not every arithmetization strategy which works for PA (or $I\Sigma_1$) also works for $I\Delta_0$

Examples

(i) the *division function* $x/y = z$ is defined by the formula

$$\varphi(x, y, z) \Leftrightarrow (y \cdot z \leq x \wedge x < y(z+1)) \vee (y = 0 \wedge z = 0)$$

Both the existence and the uniqueness of such z can be proved in $I\Delta_0$, the first by induction on $(\exists z \leq x)\varphi(x, y, z)$, the second using restricted subtraction and distribution.

(ii) the *remainder* is defined by $(x \bmod y = x \dot{-} y \cdot (x/y))$

(iii) the *division relation* $x|y$ is defined by $(x \bmod y = 0)$

(iv) the set of *primes* is defined by the formula

$$x > 1 \wedge (\forall y \leq x)(y|x \rightarrow y = x \vee y = 1)$$

The *LenBit* function

The function $LenBit(i, x)$ equals the i-th bit in the binary expansion of x and is defined by the formula $\lfloor x/i \rfloor \bmod 2$. We will use it only when $LenBit(2^i, x)$.

Example

Take $x = 5 = (1, 0, 1)$, then

$$
\begin{aligned}
LenBit(2^0, 5) &= \lfloor 5/1 \rfloor \bmod 2 = 1 \\
LenBit(2^1, 5) &= \lfloor 5/2 \rfloor \bmod 2 = 0 \\
LenBit(2^2, 5) &= \lfloor 5/4 \rfloor \bmod 2 = 1 \\
LenBit(2^3, 5) &= \lfloor 5/8 \rfloor \bmod 2 = 0 \\
\ldots
\end{aligned}
$$

A theorem on binary representation

$I\Delta_0$ can prove that the binary representation of a number uniquely defines that number:

Theorem

$I\Delta_0$ proves that $(\forall x)(\forall y < x)(\exists 2^i)(LenBit(2^i, x) > LenBit(2^i, y))$

(if we have 2 distinct numbers then there is a bit in their binary representation on which they differ)

The bootstrapping - overview

- the most important and nontrivial prerequisite of coding sequences is to define the relation $x = 2^y$
- this can be done by a Δ_0 formula $\varphi(x, y)$ and it can be shown in $I\Delta_0$ that this formula behaves as if it defined the graph of the exponentiation function with the exception that $I\Delta_0$ does not prove $(\forall x)(\exists y)\varphi(x, y)$
- the next step is to Σ_1-define Gödel numbers of sequences and the function $\beta(i, x)$ that extracts the number in the i-th entry of the sequence coded by x - this is also rather delicate

Relationships amongst the axioms of PA

Theorem

1. $B\Pi_n \vdash B\Sigma_{n+1}$
2. $I\Sigma_{n+1} \vdash B\Sigma_{n+1}$
3. *If* $A(x, \vec{y}) \in \Sigma_n$ *and* t *is a term, then* $B\Sigma_n$ *can prove that* $(\forall x \leq t)A(x, \vec{y})$ *is equivalent to a* Σ_n *formula*

To prove this theorem we use concepts that were earlier shown to be Σ_1-definable in $I\Delta_0$.

Proof - case 1

- suppose $A(x,y)$ is in Σ_{n+1}, we want to show that the following formula is derivable in $B\Pi_n$:

$$(\forall x \leq u)(\exists y)A(x,y) \rightarrow (\exists v)(\forall x \leq u)(\exists y \leq v)A(x,y)$$

- $A(x,y)$ has the form $(\exists \vec{z})B(x,y,\vec{z})$ for some $B \in \Pi_n$.
- replace the part $[\ldots(\exists y)(\exists \vec{z})B\ldots]$ by $[\ldots(\exists w)B\ldots]$, where w is intended to range over the codes of the Gödel numbers of sequences of possible values for y and $\vec{z}$ by setting

$$\beta(1,w) = y \text{ and } \beta(i+1,w) = z_i$$

- since $y = \beta(1,w) < w$, take w to witness the bound for y in the consequent of the above axiom

Proof - case 3 (this is needed for case 2)

- by induction on n, if $n = 0$, then the new formula is bounded in $I\Delta_0 \subseteq B\Sigma_0$
- since we can code a sequence of possible values by a single number, let A is of the form $(\exists y)B$ for some $B \in \Pi_{n-1}$, then

$$\begin{aligned}(\forall x \leq t)(\exists y)B &\Leftrightarrow (\exists u)(\forall x \leq t)(\exists y \leq u)B && \text{(REPL)}\\ &\Leftrightarrow (\exists u)(\forall x \leq t)C && \text{(IH)}\end{aligned}$$

where C is Π_{n-1}, so $(\forall x \leq t)A$ is equivalent to a Σ_n formula

Proof - case 2

Suppose $A(x, y) \in \Sigma_{n+1}$, we want to show that $I\Sigma_{n+1}$ proves the REPL instance for A, by case 1 we may assume that $A \in \Pi_n$.

- assume
$$(\forall x \leq u)(\exists y)A(x, y) \tag{1}$$
- denote by $\varphi(a)$ the formula
$$(\exists v)(\forall x \leq a)(\exists y \leq v)A(x, y) \tag{2}$$
- note that $\varphi(x)$ is equivalent to a Σ_{n+1} formula (case 3)
- by (1) we have $\varphi(0)$ and $\varphi(a) \to \varphi(a + 1)$ for $a < u$
- so by Σ_{n+1}-induction it holds that $\varphi(u)$

Some other relationships

(i) $I\Sigma_n \vdash I\Pi_n$

- let $A(x) \in \Pi_n$, assume $A(0)$ and $(\forall x)(A(x) \rightarrow A(x+1))$
- let a be arbitrary, let $B(x)$ be the formula $\neg A(a \dot{-} x)$
- then $\neg B(a)$ and $B(x) \rightarrow B(x+1)$, so by induction $\neg B(0)$
- hence $A(a)$, and therefore also $(\forall x)A(x)$

(ii) $I\Pi_n \vdash I\Sigma_n$ is similar

Some other relationships

(iii) $L\Sigma_n \vdash I\Pi_n$
- take $A(x) \in \Pi_n$ such that $(\exists x)\neg A(x)$
- use LNP to find the smallest x' such that $\neg A(x')$
- if $x' = 0$, then $\neg A(0)$
- if $x' > 0$, then by LNP $A(x' - 1)$

(iv) $L\Pi_n \vdash I\Sigma_n$ is similar

(v) ... and IND also implies LNP

Some arrows

$$
\begin{array}{l}
I\Sigma_{n+1} \\
\Downarrow \\
B\Sigma_{n+1} \Leftrightarrow B\Pi_n \\
\Downarrow \\
I\Sigma_n \Leftrightarrow \Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n
\end{array}
$$