The fusion method (AKA the ultraproduct)

Ondřej Ježil

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  - Razborov's approximation method
  - Sipser's topological approach
- These approaches were unified by M. Karchmer with his "Fusion method"
- we will cover the survey article: Avi Widgerson – The Fusion Method for Lower Bounds in Circuit Complexity

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- Fusing computations
  - We have some program P, accepting exactly $U \subseteq \{0,1\}^n$, and for each $u \in U$, we have $P(u)$ an accepting computation of u.
  - If we have some finite analogue of an ultrafilter F, we can fuse them into a new "accepting computation" of some new z, a contradiction.

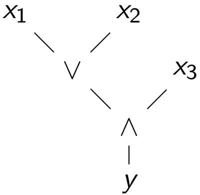
Straight-line programs, computations

Definition

Let $X = \{x_1, \ldots, x_k\}$ be a set of variables. A ***straight-line program*** *P is a tuple $(g_1, \ldots, g_t)$, such that $g_i = x_i$ for $i \in \{0, \ldots, n\}$ and $g_i = g_{i_1} \circ_i g_{i_2}$ where $i_1, i_2 < i$, and $\circ_i \in OP$ some set of binary operations.*
For $u \in \{0, 1\}^n$ we define a ***computation*** *of P on input u as $P(u) := (g_1(u), \ldots, g_t(u))$, where $g_t(u) \in \{0, 1\}$ is the output of the computation.*

An example of a straight-line program

- Consider the circuit:



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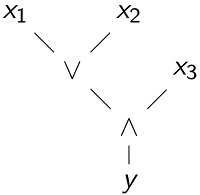

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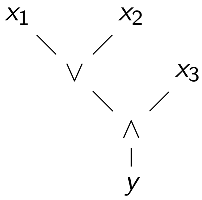

- The corresponding straight-line program is

$$P = (x_1, x_2, x_3, x_1 \vee x_2, (x_1 \vee x_2) \wedge x_3).$$

- And the following is an accepting computation of $P(1, 0, 1)$

$$P(1, 0, 1) = (1, 0, 1, 1, 1).$$

The fusion method

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- This is equivalent to finding a lower bound for a straight-line program computing some boolean function f on n-letter strings by setting $U = f^{-1}[1]$.
- Assume for contradiction there exists some program $P = (g_1, \ldots, g_t)$ that accepts exactly U and t is too small.

The accepting computation matrix

- Consider a $|U| \times t$ matrix, where rows are indexed by U and each row is equal to the computation $P(u)$.

u					the rest of $P(u)$				
0	1	...	0	1	0	1	...	0	1
0	0	...	1	1	0	0	...	0	1
1	0	...	0	1	1	0	...	1	1
1	0	...	1	0	1	0	...	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	...	1	0	0	0	...	0	1

Producing a contradiction

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1	0	...	1	0	1	0	...	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	...	1	0	0	0	...	0	1

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1	1	...	1	0	0	0	...	0	1

- We would like to produce a contradiction using that the number of rows t is too small.
- We will try to construct a "new" accepting computation using the old ones. Since this table contains all accepting computations, this would be a contradiction.

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- Let $F\colon \{0,1\}^{|U|} \to \{0,1\}$, "a functional" from some set Ω of functionals (will be specified later, e.g. $\Omega = \{\text{all functionals}\}$ works).
- F will act as our finite analogue of an ultrafilter.

Applying the functional

0	1	...	0	1	0	1	...	0	1
0	0	...	1	1	0	0	...	0	1
1	0	...	0	1	1	0	...	1	1
1	0	...	1	0	1	0	...	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	1	...	1	0	0	0	...	0	1

$\downarrow_F$

0	

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0	1	...	1	1	1	0	...	1	1

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$$\Omega_f = \{F \in \Omega; F \text{ satisfies the first two points}\}.$$

- How do we find functional in Ω_f that satisfies the third requirement, since it depends on P?
- We don't! We just conclude that if such short P exists, there has to be no such functional in Ω_f.

Covering

Definition

Let OP be some set of operations. We say, that the triple $(g, h, \circ)$, $g, h \in \{0,1\}^n \to \{0,1\}, \circ \in OP$ ***covers*** *a functional F, if*

$$F(g) \circ F(h) \neq F(g \circ h).$$

For a function $f\colon \{0,1\}^n \to \{0,1\}$ *we denote* $\rho(f)$ *the smallest number of such triples that cover* Ω_f.

The lower bound

Theorem (Meta-theorem)

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- The lower bound is actually $n + \rho(f)$.
- We can restrict the smallest cover to those covers for which each $(g, h, \circ)$ has g, h definable by some straight line program over OP.

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- The last column contains only ones because of requirement 2.

Example - parity cont.

- We need to cover the following four functionals.

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- F_3 is covered by $(x_1, -, \neg)$, since $\neg F_3(x_1) = 1$, but $F_3(\neg x_1) = F_3(x_2) = 0$ and so is F_4
- This is the smallest possible cover using OP, therefore the lower bound is $2 + 3 = 5$.

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  - While considering unrestricted Ω we can obtain a larger lower bound. However in some situations for some restrictions we get the following theorem:

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Proof.

(sketch) We have a cover $C = \{(g_1, h_1, \circ_1), \ldots, (g_t, h_t, \circ_t)\}$.

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With the claim, we just need to construct a program, that tries to find such F. We don't need the whole functional, just its values on x_i and the cover. For many choices of OP and Ω this yields program, that has either linear or polynomial length with respect to $\rho(f)$. □

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- What we've seen so far is actually his "generalized approximation method", in this point of view, F is seen as an approximation of a gate.
- 1990 Razborov proved that somewhat restricted can be associeted with non-deterministic branching programs, an proved a super-linear lower bound for the Majority function.

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- In his 1984 paper Sipser asks for a finite analogue of a limit that will allow us to carry out such arguments in the finite world.
- This should remind us of Ω a finite notion of a limit, and F a notion of a converging sequence.

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- He noted, that this method can be viewed as a finitary version of an ultraproduct. This idea was pushed even further by Ben-David, Karchmer and Kushilevitz who have showed that standard ultra-filter arguments can simplify Sipser's proof.

The unification - Karchmer's work cont. 1

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Theorem (Characterization of $\mathbf{P}$)

$f \in \boldsymbol{P}$ if and only if $\rho(\Omega_f) \leq p(n)$ for some polynomial p.

The unification - Karchmer's work cont. 2

- Now consider: $\Omega' := \{F; F \text{ a self dual filter}\}$, that is a set of filters, that contain each string or its negation.

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- Note that Ω' contains more than just ultrafilters.
- We have the following result:

Theorem (Characterization of NP)

$f \in \textbf{NP}$ if and only if $\rho(\Omega'_f) \leq p(n)$ for some polynomial p.

The unification - Karchmer's work cont. 3

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Theorem (Characterization of $m\mathbf{P}$)

$f \in m\mathbf{P}$ if and only if $\rho_+(\Omega_f) \leq p(n)$ for some polynomial p.

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Theorem (Characterization of $m\mathbf{P}$)

$f \in m\mathbf{P}$ if and only if $\rho_+(\Omega_f) \leq p(n)$ for some polynomial p.

- Karchmer used this to give a new proof of Razborov's super-polynomial lower bound for the monotone clique.

Algebraic variants

- $(\{0,1\}^n, \wedge, \vee, (\neg))$ are precisely finite Boolean algebras, and a filter is a natural notion for these structures, that can give some intuition on the choice $\Omega = \{\text{filters}\}$

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- Notice, that the whole fusion method does not depend on that the values of our functions are just $\{0,1\}$, if instead we consider functions over some ring R, this whole method works for proving lower bound on their algebraic circuit complexity.

Table of results

Inputs	Gates	Type	Mode	Ω	$\mathcal{C}_\Delta$	Upper bound
$X \cup \overline{X}$	$\{\vee, \wedge\}$	Circuit	Det.	Filters	**P**	$(\rho_\Gamma(f))^C$
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- For *m***P** there exists a super-polynomial lower bound: $\rho_\Gamma(f) = \exp(\Omega(n^{1/8}))$