Determinacy

Andrés Eduardo Caicedo

Department of Mathematics
Boise State University

Graduate Student Seminar, September 8, 2010

Thanks to the NSF for partial support through grant DMS-0801189.

My work is in set theory, and lately I have been looking at some problems in the area of determinacy. This talk will introduce the basic concepts in this theory.

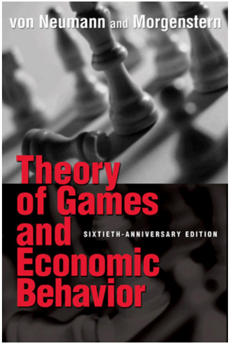
Some games

Trivia

The first known occurrence of what we recognize as game theory is in a letter by James Waldegrave dated 1713.

Earl Waldegrave discussed a card game known as Le Her. Its study led to some early developments in probability theory.

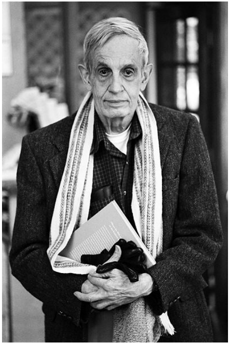
The earliest writings on game theory focused on specific games of chance and were quickly identified as relevant to economics. It is in this framework that mathematicians usually think of games, and John von Neumann is considered the founder of the theory, with a series of papers in 1928 and a famous treatise coauthored with Oskar Morgenstern, in 1944.

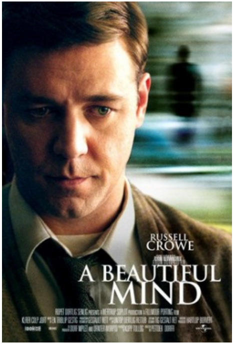


It is in this context that his famous Minimax theorem was proved.

> "As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved"

Equally famous in this context is the concept of Nash equilibrium, due to John Nash.

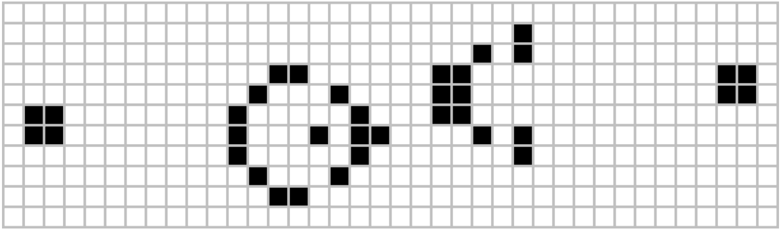




This is not the line of study I want to discuss. Rather, I want to say a few words about the "combinatorial" theory of *perfect information* games, where chance is not allowed. In particular, I will focus on games of infinite length between two players, who alternate their moves. I will mostly be interested in games where one of the players has a winning strategy.

This leaves aside many interesting examples (besides games of chance), such as:

- Zero-player games. Typical examples are cellular automatons, in particular, John Conway's game of life.



- One-player games. These are commonly referred to as puzzles.

							1	
4								
	2							
				5		4		7
		8				3		
		1		9				
3			4			2		
	5		1					
			8		6			

(This is a Sudoku puzzle with 17 initial entries. It is open whether one can have one with at most 16 entries and unique solution.)

- Three or more-player games. The problem here is that there are very trivial examples where no player has a winning strategy (essentially, the rock-paper-scissors game).

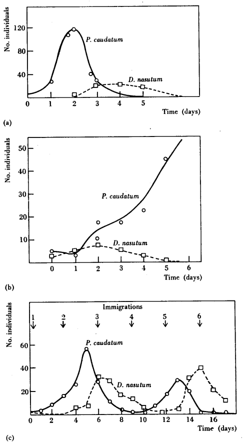
(Found at http://www.culch.ie/)

For example, consider the game where players I, II, and III play 0 or 1, with I playing first the number n_I, then II the number n_{II}, and finally III the number n_{III}. The winner is specified as follows:

- Player I wins iff $n_{II} \neq n_{III}$.
- Player II wins iff $n_{II} \neq n_I$ and $n_{II} = n_{III}$.
- Player III wins iff $n_I = n_{II} = n_{III}$.

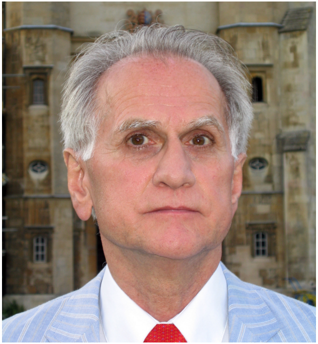
- Continuous games.

Typical examples are pursuit-evasion (predator-prey games), usually modeled in terms of differential equations.



(From http://www.globalchange.umich.edu/globalchange1/current/lectures/predation/predation.html.)

Continuous games allow for a very curious possibility: There are games between two players where *both* players have a winning strategy. This was recently noticed by Béla Bollobás, Imre Leader, and Mark Walters.



(Leader is a well known Othello player. His picture is from the Flickr page of Svenska Othelloforbundet. I could not find a picture of Walters.)

Briefly, I want to explain how this is possible. The example by Bollobás, Leader, and Walters is a "lion-man pursuit and evasion game".

From their paper "Lion and Man—Can both win?":

> Rado's famous 'Lion and Man'problem ... is as follows. A lion and a man (each viewed as a single point) in a closed disc have equal maximum speeds; can the lion catch the man?

We generalize by changing the disk with a metric space (X, d).

A run of the game takes place in time, modeled by $[0, 1]$. The moves of both players are points in X, so the final object a player produces is a function $f : [0, 1] \to X$. We specify initial points $x_l \neq x_m$ where the lion and the man must begin at time $t = 0$.

We model that the lion and the man have equal maximum speed by imposing the condition that f is *Lipschitz with constant 1*, i.e.,

$$d(f(a), f(b)) \leq |a - b|$$

for any $a, b \in [0, 1]$.

Say the lion is playing f_l and the man is playing f_m. For any $t \in [0, 1]$, to choose $f_l(t)$, the lion has access to $f_m \restriction [0, t)$ and similarly for the man.

The lion wins if it catches the man, and the man wins otherwise. This means that the lion wins iff for some $t \in [0, 1]$, $f_l(t) = f_m(t)$.

A strategy for the lion is a function Φ that to each path m by the man assigns a path $\Phi(m)$ by the lion in such a way that if m_1, m_2 coincide on $[0, t)$, then $\Phi(m_1)$ and $\Phi(m_2)$ agree on $[0, t]$.

To say that the strategy Φ is *winning* simply means that for every path m by the man there is a t such that $\Phi(m)(t) = m(t)$.

(Winning) strategies for the man are defined analogously.

The l_∞ *sum* of two metric spaces (X, d_X) and (Y, d_Y) is the metric space (Z, d) where $Z = X \times Y$ and

$$d((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

Theorem (Bollobás-Leader-Walters, 2009)

Let X be the l_∞ sum of the closed unit disc D and $[0, 1]$. Then, in the lion and man game on X with the man starting at $((0, 0), 0)$ and the lion starting at $((0, 0), 1)$, both players have winning strategies.

This seems impossible. Because if m and l are the paths the man and the lion play following their respective strategies, then $m(t) \neq l(t)$ for all t, since the man's strategy is winning, and $m(t_0) = l(t_0)$ for some t_0, since the lion's strategy is winning.

The point is that there is no way of having both the man and the lion use their strategies against the other. More precisely, if Φ is the lion's strategy, and Ψ is the man's, we would need to have $\Phi(m) = l$ and $\Psi(l) = m$. And, for the space X of the theorem, one can explicitly produce strategies Φ and Ψ without such *fixed points*.

This feature is unique of continuous games, and there is probably much more that one can do in this setting.

Perfect information games

In set theory, we study "discrete" games, formalized as follows: We fix a set X; the elements of X are the possible moves.

Two players, I and II, alternate, with I playing first. The rules of each particular game determine which moves are valid. In each turn, the corresponding player picks an element of X that is a valid move; if there is no such element, the player loses. If the game continues for infinitely many moves, the winner is decided depending on the sequence in $X^{\mathbb{N}}$ that has been produced.

Both players know the rules of the game, have access to the elements of X and, at each turn, know what moves have been made so far. We call these perfect information games.

Finite games

I	x_0		x_2		$\dots$
II		x_1		x_3	

If X is finite and any run of the game always ends after a finite number of moves, we say that the game is *finite*.

Typical examples are tic-tac-toe and chess. Games of chance do not fall within this framework.

Another restriction is that games cannot end in a tie. This would eliminate chess from our analysis, but we can include it if we decree that, say, black "wins" if the game ends up tied.

Formally, a *strategy* is a function that assigns to each finite sequence of elements of X an element of X,

$$\sigma : X^{<\mathbb{N}} \to X.$$

We say that I *follows* the strategy σ if each move of I is dictated by σ and the previous moves of II:

I	$\sigma(\langle\rangle)$		$\sigma(\langle x_1 \rangle)$		$\sigma(\langle x_1, x_3 \rangle)$		
II		x_1		x_3			$\dots$

Similarly, we can talk of II *following* σ.

We say that σ is a *winning strategy* for I in a game $\eth$ if I wins whenever I follows σ. (Similarly for II.) A game is *determined* if one of the players has a winning strategy.

The mathematical analysis of games began with Ernst Zermelo (1913), with chess being the game originally singled out. From Zermelo's arguments it follows that all finite games are determined.

Bengt Ekeroth and Max von Sydow in *The Seventh Seal*, 1957. © Svensk Filmindustri; Ingmar Bergman (director), Gunnar Fischer (cinematography).

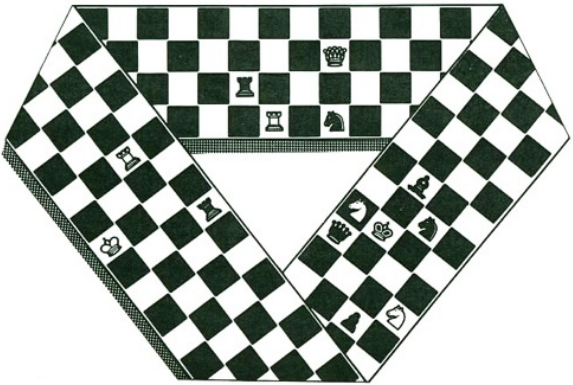
The idea can be explained briefly. Fix a game $\mathfrak{D}$.

Suppose the first player does not have a winning strategy. This means that for every move x_0 by player I there is a move $x_1 = \sigma(\langle x_0 \rangle)$ by player II such that player I still does not have a winning strategy in the game $\mathfrak{D}_{x_0 \frown x_1}$ starting from this position $x_0 \frown x_1$. Otherwise, for some x_0 and any x_1, player I has a winning strategy τ_{x_1} in $\mathfrak{D}_{x_0 \frown x_1}$. But then, first playing x_0 and then using τ_{x_1} would be a winning strategy for I after all.

Similarly, for any x_2, there is an $x_3 = \sigma(\langle x_0, x_2 \rangle)$ such that player I does not have a winning strategy in the game starting at position $x_0 \frown x_1 \frown x_2 \frown x_3$. Etc. This defines a strategy σ for player II.

Since the game is finite, and we just ensured that player I cannot win the game at any given move (because then I would certainly have a—trivial—winning strategy when reaching this winning position), this means that this strategy is winning for II.

Infinite games



(Found at http://www.ocf.berkeley.edu/∼wwu/)

David Gale and F. M. Stewart noticed that the argument above readily generalizes to show that open games are determined.

A game $\Game$ is *open* for player I if, whenever I wins, this had already been decided after finitely many moves.

Given $A \subseteq X^{\mathbb{N}}$, the game $\Game_X(A)$ is defined so that there are no restrictions on what elements of X can be played at each turn, and I wins iff the resulting sequence $x = \langle x_0, x_1, \dots \rangle$ is an element of A.

Theorem (Gale-Stewart, 1953)

Suppose X is given the discrete topology, and $X^{\mathbb{N}}$ the product topology. If A is an open subset of $X^{\mathbb{N}}$, then $\eth_X(A)$ is determined.

A careful inductive argument generalized this as follows:

Theorem (Martin, 1975)

If A is a Borel subset of $X^{\mathbb{N}}$, then $\eth_X(A)$ is determined.

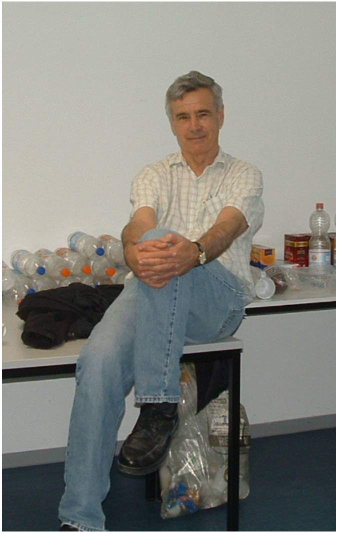
Martin's result is significant in several respects. For example, even if $X = \{0, 1\}$, the result *cannot* be proved without using sets of very large size: larger than $\mathbb{R}$, $2^{\mathbb{R}}$, $2^{2^{\mathbb{R}}}$, etc.

Using the theory of large cardinals, one can extend the analysis when $X = \mathbb{N}$ much further:

Theorem (Martin-Steel, Woodin, 1985)

If $A \in L(\mathbb{R})$, then $\Game_{\mathbb{N}}(A)$ is determined.

The class $L(\mathbb{R})$ is a model of set theory without choice; it contains every subset of $\mathbb{N}^{\mathbb{N}}$ that is reasonably definable. Certainly, every Borel set, and every set that ever appears in Analysis, gives us a determined game, because there is a natural homeomorphism $\mathbb{N}^{\mathbb{N}} \simeq \mathbb{R} \setminus \mathbb{Q}$.



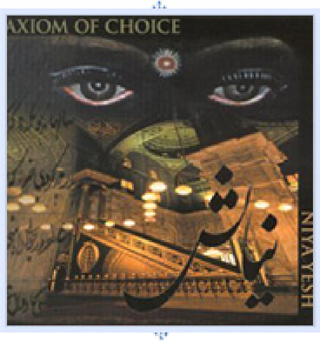
The axiom of choice

On the other hand, we cannot hope that every game is determined:

Fact

There is a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ with $\Game_{\mathbb{N}}(A)$ undetermined.

This uses in an essential way the axiom of choice.



The axiom of choice is part of the basic set of axioms of set theory. One of its most popular formulations is that every set can be well-ordered.

The basic set of axioms of set theory is ZFC. The theory that results when choice is not included is denoted ZF. It was introduced by Zermelo and Abraham Fraenkel.

One can prove that choice is equivalent to Tychonoff's theorem, to Zorn's lemma, to the fact that every vector spaces admits a basis, etc.

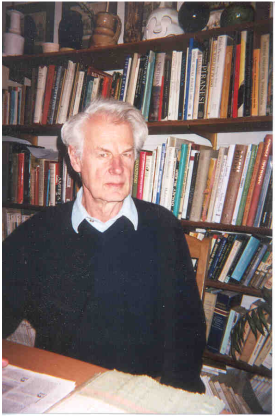
Choice is also equivalent to the Gale-Stewart theorem that open games are determined!

If X is well-ordered, the games $\Game_X(A)$ with A open are determined, and the proof does not require choice.

On the other hand, working in ZF, one can show that there is an undetermined game $\Game_{\omega_1}(A)$. Here, ω_1 is the first uncountable well-ordered set.

Mudvayne. *Determined*, in *Lost and Found* (2005) Epic Records.

The *Axiom of determinacy*, suggested by Jan Mycielski and Hugo Steinhaus in 1962, states that all games $\Game_{\mathbb{N}}(A)$ are determined. The fact above indicates that this axiom is inconsistent with the usual system of axioms of set theory, ZFC. The result of Martin-Steel and Woodin mentioned before indicates that it is consistent with ZF, the usual system excluding the axiom of choice.


PIRANESI

Since we accept choice, determinacy is just false. However, the models of determinacy have a very rich structure that "carries over" to the universe of sets, where choice holds.

One of the consequences of determinacy is the perfect set property, a version of the *continuum hypothesis*: If $A \subseteq \mathbb{N}^{\mathbb{N}}$ then either A is countable, or else it contains a perfect subset. Since determinacy holds in $L(\mathbb{R})$, and $\mathbb{N}^{\mathbb{N}}$ can be identified with the irrationals, it follows that any set of reals that appears in analysis also has this property.

Similarly, determinacy gives us that every set of reals is Lebesgue measurable and has the property of Baire.

Large cardinals

This highlights one of the advantages of studying determinacy: It provides us with a unifying framework to explain why “natural” sets of reals are well-behaved.

This analysis makes essential use of large cardinals.

Cardinals are just the sizes of sets. Central to the very existence of set theory is that we can make sense of different sizes of infinity.

Large cardinals are an essential part of modern set theory. These are the sizes of sets that are so large that their existence *cannot* be proved in ZFC.

They are useful because they provides us with elementary embeddings, which provide the universe of sets with a rich structure—akin to the automorphisms of a group. The problem is that the universe of sets has *no* automorphisms.

Many people have contributed to the study of large cardinals. I'll just mention Robert Solovay, who was the first to note that determinacy implies the existence of large cardinals. This is a deep connection that is still being explored.

Multiboard games

Marion Scheepers and I have been looking, together with Fred Galvin and Richard Ketchersid, at some results about the determinacy of a slightly different kind of games.

In the 1970s, Galvin suggested an approach to the study of undetermined games $\Game_X(A)$. One fixes a cardinal κ. In the game $\Game_X(A, \kappa)$, rather than playing in one board, players I and II compete simultaneously on κ many boards. Player I wins iff there is some board in which I wins the corresponding run of $\Game_X(A)$.

(Found at http://worldrecordsacademy.org)

Given $A \subseteq X^{\mathbb{N}}$ there are three possibilities:

- Player II has a winning strategy in $\eth_X(A)$ and therefore in $\eth_X(A, \kappa)$ for all κ.
- There is some κ_0 such that $\eth_X(A, \lambda)$ is undetermined if $\lambda < \kappa_0$ but player I has a winning strategy if $\lambda \geq \kappa_0$.
- $\eth_X(A, \kappa)$ is undetermined for all κ.

Galvin obtained results and formulated questions about these games, but they remain unpublished and mostly unknown.

Galvin, Ketchersid, Scheepers, and I are (finally) writing a survey on this topic.

Perhaps somewhat surprisingly, given that the determinacy of all games $\mathfrak{D}_{\mathbb{N}}(A)$ contradicts choice and that, even without choice, there are undetermined games $\mathfrak{D}_{\omega_1}(A)$, we have:

Theorem

It is consistent that for all X and all $A \subseteq X^{\mathbb{N}}$, there is some κ such that $\mathfrak{D}_X(A, \kappa)$ is determined.

This is an interesting feature of modern mathematics: Statements can be true or false, or *neither*, and establishing that they are consistent may involve using large cardinals in an essential way.

A dichotomy

Much work remains to do in the area of determinacy. Ketchersid and I have recently shown several results studying the structure of cardinals in *natural* models of determinacy. Let me close with one of our results. Recall that determinacy contradicts choice. In fact, $\mathbb{R}$ is an example of a non-well-orderable set if determinacy holds.

Theorem

In natural models of determinacy, any set is either well-orderable, or else contains a copy of the reals.

(Found at http://anonymusings.blogsome.com)

The end.