Approximate Euler characteristic, dimension, and weak pigeonhole principles

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Abstract

We define the notion of approximate Euler characteristic of definable sets of a first order structure. We show that a structure admits a non-trivial approximate Euler characteristic if it satisfies weak pigeonhole principle WPHP_n^{2n}: two disjoint copies of a non-empty definable set A cannot be definably embedded into A, and principle CC of comparing cardinalities: for any two definable sets A, B either A definably embeds in B or vice versa. Also, a structure admitting a non-trivial approximate Euler characteristic must satisfy WPHP_n^{2n}.

Further we show that a structure admits a non-trivial dimension function on definable sets iff it satisfies weak pigeonhole principle $\mathrm{WPHP}_n^{n^2}$: for no definable set A with more than one element can A^2 definably embed into A.

An abstract Euler characteristic (Ec) on a first order structure assigns to definable sets values in a commutative ring such that basic properties of counting with finite sets are fulfilled (we recall the formal definition from [6] in Section 2). Not all structures admit nontrivial Ec. For example, if there is a definable bijection between a definable set and the set plus one other point (i.e., the so called onto-pigeonhole principle ontoPHP fails), then $0 = 1$ in the ring and everything is trivial. In fact, the validity of the ontoPHP characterizes structures admitting the weak Ec, cf.[6] or Theorem 2.2. The

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ordinary PHP: a definable set cannot be definably embedded into its proper subset, characterizes structures where the weak Ec is partially ordered, cf.[9].

Important variants of the pigeonhole principle are the so called weak pigeonhole principles; while PHP principles say in some form or other that a bigger set cannot embed into a smaller one, WPHP principles assert that a set cannot embed into a much smaller set. The particular definition of the qualification *much* determines various forms of WPHP. WPHP principles are prominent in proof complexity and in bounded arithmetic, a bridge between complexity theory and logic. One motivation for the current research is the collection of open problems about the provability of WPHPs in weak formal systems, cf.[5]. These problems amount to constructions of models (of various weak arithmetics) in which WPHPs fail, and one wants to understand general properties of such models. Another motivation is somewhat more general. In [1, 6, 9, 7] it was shown that the validity of forms of PHP and of some other counting principles (e.g. the modular counting principles, cf.[1, 6]) in a structure is equivalent to the existence of an abstract Euler characteristic for sets definable in the structure with particular properties (depending on the particular counting principles). Hence it is interesting to find out if WPHPs can be also characterized by the existence of some natural invariants of definable sets.

We shall consider two variants of WPHP:

- WPHP_n^{2n}: Two disjoint copies of a non-empty definable set A cannot be definably embedded into A.
- $\mathrm{WPHP}_n^{n^2}$: For no definable set A with more than one element can A^2 definably embed into A.

It will not be difficult to see that $\mathrm{WPHP}_n^{n^2}$ principle characterizes structures that admit a non-trivial dimension function (in the sense of Schanuel [12]; we shall recall the definition in Section 3) on their definable sets. The case of WPHP_n^{2n} is more involved and requires an introduction of a new invariant of definable sets which we shall call *approximate Euler characteristic* (aEc).

The idea of aEc is that similarly as Ec formalizes counting, aEc will formalize approximate counting: aEc counts the size of definable sets but with a possible error, the error being a negligible percentage. This is an important issue in complexity theory (and in bounded arithmetic) as for some complexity classes the exact counting is hard (e.g., given a polynomial-time set the function - of x - that counts the number of elements in the set below x is generally assumed not to be polynomial-time) while an approximate counting (the function counts the number of elements with some bounded error) is possibly much simpler. We shall show that a structure admits a non-trivial approximate Euler characteristic if it satisfies weak pigeonhole principle WPHP^{2n}_n and the principle CC of comparing cardinalities (this principle and its two variants were considered in [9]):

- *CC*: For any two definable sets A, B either A definably embeds in B or vice versa.

On the other hand, a structure admitting a non-trivial approximate Euler characteristic must satisfy WPHP^{2n}_n.

One would like to avoid using the CC principle in the construction. That would then give an exact characterization of structures with non-trivial aEc as those satisfying WPHP^{2n}_n. CC could be avoided, in principle, if it would hold that any structure satisfying WPHP^{2n}_n can be expanded to a structure still satisfying WPHP^{2n}_n but also satisfying CC. We do not know if this is true.

The paper is organized as follows. In Section 1 we give few preliminaries on semirings. Section 2 recalls definitions of (ordered) Ec and facts about them from [6, 9]. The dimension function and $\mathrm{WPHP}^{n^2}_n$ are considered in Section 3. The definition of aEc and the connection to WPHP^{2n}_n are in Section 4. Section 5 offers several examples of structures and their aEc'. More background information can be found in [6, 9, 7].

1 Preliminaries on semirings

A semiring, or a rig in an equivalent terminology, is a structure $(R, 0, 1, +, \cdot)$ having properties like a ring, except that $(R, 0, +)$ may be only a semigroup. We shall consider only commutative semirings. Examples are $\mathbf{N}$, $\mathbf{N}[x]$ or generally ordered rings without the "negatives"[1].

A semiring important for us is constructed from sets definable in a structure M as follows. Let $\mathrm{Def}(M)$ be the collection of subsets of all M^k, $k \geq 1$, that are definable in M with parameters. Two definable sets A, B are equivalent, $A \sim B$, iff there is a definable bijection between them. The set $\widetilde{\mathrm{Def}}(M) := \mathrm{Def}(M)/\sim$, together with zero $0 := \emptyset/\sim$, one $1 := \{a\}/\sim$ (a any element), and operations $+$ - disjoint union, and $\cdot$ - Cartesian product

[1] Hence the name rig, e.g. in [12]. A full definition can be found there.

forms a semiring. We shall denote the quotient map from $\mathrm{Def}(M)$ to $\widetilde{\mathrm{Def}(M)}$ by $[\ldots]$.

A partially ordered ring R is R together with a subset $P \subseteq R$ such that $0, 1 \in P$, $P + P \subseteq P$, $P \cdot P \subseteq P$, and if $0 \neq x \in P$ then $-x \notin P$. P are the non-negative elements of R and the partial ordering $x \leq y$ is given by the condition $y - x \in P$.

For a an element of a semiring and $k \in \mathbf{N}$, $k \cdot a$ denotes the scalar multiple $a + \ldots + a$, k-times.

Definition 1.1 *Let $(S, 0, \oplus, \leq)$ be a linearly ordered commutative semigroup and R a linearly ordered semiring.*

The linearly ordered commutative semiring $R[x^S]$ consists of elements that are finite R-linear combinations of expressions x^u, $u \in S$. Formally, $R[x^S]$ is the set of partial finite functions $f : S \to R$ with the addition defined coordinate-wise and the multiplication defined by the rule: $(f \cdot g)(s) = \sum_{u+v=s} f(u) \cdot g(v)$. The linear ordering is the lexicographic ordering given by: $f < g$ iff $f(u) <_R g(u)$, where $u \in S$ is the maximal (in $<_S$) in the domains of both f and g such that $f(u) \neq g(u)$.

Note that $R[x^S]$ is a ring if R is a ring. We shall use this construction for R one of $\mathbf{N}$ (natural numbers), $\mathbf{Q}$ (rationals) or $\mathbf{R}$ (reals), and S being the additive semigroup of the non-negative reals $(\mathbf{R}^{\geq 0}, 0, +, \leq)$.

2 Euler characteristic

The abstract Euler characteristic on first-order structures has been defined in [6]. Its variant, the weak Ec, is a special case of a general construction by Schanuel [12]. He defined (in our terminology) the weak Euler characteristic and the dimension function on any distributive category. However, I shall follow [6, 9] and consider only the category of sets and maps definable in a structure.

Definition 2.1 ([6, Def.2.1]) *Let M be a first-order structure. Let R be a commutative ring with unity. A function*

$$\chi : \mathit{Def}(M) \longrightarrow R$$

is an Euler characteristic on M over R *iff it satisfies the following conditions: $\chi(\{a\}) = 1$ for any $a \in M^k$, χ is additive on disjoint unions and* *multiplicative on Cartesian products, and χ factors through the quotient map* $[\ldots] : \mathit{Def}(M) \to \widetilde{\mathit{Def}(M)}$.

The last condition is that

- $\chi(A) = c \cdot \chi(B)$, *whenever* $c \in R$, $A, B \in \mathit{Def}(M)$ *and there is a definable map* f *with domain* A *and range* B *such that each its fiber* $f^{(-1)}(b)$, $b \in B$, *has Euler characteristic* $\chi(f^{(-1)}(b)) = c$.

The function χ/R *is a weak Euler characteristic iff* χ/R *satisfies all conditions but the last one. Euler characteristic is sometimes called strong Euler characteristic, in order to distinguish it from the weak one.*

One can state the definition of weak Ec in a more algebraic language: The weak Ec is the quotient map $[\ldots] : \mathrm{Def}(M) \to \widetilde{\mathrm{Def}(M)}$ composed with a semiring-homomorphism of $\widetilde{\mathrm{Def}(M)}$ into a ring. We use the definition explicitly referring to $\mathrm{Def}(M)$ as it allows for the formulation of the fifth, fiber, condition. That condition is important for various applications (e.g. [6]) and it has also the nice property that it implies, together with condition 2., all other conditions, cf. [6].

If M admits a weak Ec then there exists a universal one with the target ring denoted $K_0(M)$ - the Grothendieck ring of the structure, cf. [6, 9].

The following theorem characterizes when a structure admits a weak Euler characteristic. No such transparent characterization is known for strong Ec, but there is also the universal strong Ec (see [9]).

Theorem 2.2 ([6, Thm.3.1]) *M admits a nontrivial weak Euler characteristic iff it satisfies the onto pigeonhole principle ontoPHP: There is no definable set and a definable bijection between the set plus one other point and the set.*

Definition 2.3 ([9, Def.4.2]) *A weak Euler characteristic χ/R on M is partially ordered if R is partially ordered and the range of χ is a subset of the non-negative elements of R. Equivalently, if $A \subseteq B$ are definable sets then $\chi(A) \leq \chi(B)$.*

Theorem 2.4 ([9, Thm.4.3]) *M admits an ordered weak Euler characteristic iff it satisfies the ordinary pigeonhole principle PHP: There is no definable set and a definable injection of the set into a proper subset of itself.*

3 Dimension

Dimension of definable sets (in a special case of the definition by Schanuel [12]) is a semiring-homomorphism of $\widetilde{\mathrm{Def}(M)}$ into a semiring in which $1+1=1$. For a part of the next definition we need to introduce a partial ordering on a semiring S. Define the relation $a \leq b$ by the condition $a+b=b$. If S satisfies $1+1=1$ then the relation is a partial ordering ($1+1=1$ implies the reflexivity).

Definition 3.1 ([12],[9, Sec.9]) *An abstract dimension function on M is a function*

$$d : Def(M) \to S$$

where S is a semiring in which $1+1=1$, such that d factors through $[\ ] : Def(M) \to \widetilde{Def(M)}$ and $d/\sim$ is a semiring homomorphism.

We say that dimension d is non-trivial iff $d(A^2) > d(A)$ for all infinite definable sets A.

Semirings S in which $1+1=1$ can be transparently given in the "logarithmic" notation as an upper semilattice $(S, -\infty, \leq, 0, \vee, \oplus)$ in which 0 becomes $-\infty$, $+$ becomes the semilattice union $\vee$ with partial ordering $\leq$, 1 becomes 0 and $\cdot$ becomes $\oplus$.

It will be convenient to think of $-\infty$ as an extra element of the universe of $(S, -\infty, \leq, 0, \vee, \oplus)$ outside of S. Then $(S, -\infty, \leq, 0, \vee, \oplus)$ is uniquely determined by the partially ordered semigroup $(S, 0, \oplus, \leq)$ in which 0 is the minimal element and $a \leq b$ implies $a \oplus c \leq b \oplus c$. On the other hand, having a partially ordered semigroup $(S, 0, \oplus, \leq)$ we can adjoin a new element $-\infty$, stipulate that $-\infty \oplus a = -\infty$ and $-\infty < a$ for all $a \in S$, and define $\vee$ from the ordering. In this way there is a correspondence between semirings where $1+1=1$ and partially ordered semigroups. One can restate the definition of a dimension as a function d from $\widetilde{\mathrm{Def}(M)} \setminus \{[\emptyset]\}$ into a partially ordered semigroup $(S, 0, \oplus, \leq)$ such that:

- $d([\{a\}]) = 0$,
- $d([A]) \leq d([B])$, if A definably embeds into B,
- $d([A \dot{\cup} B]) = max(d([A]), d([B]))$,
- $d([A \times B]) = d([A]) \oplus d([B])$.

Note that the condition of the non-triviality of d implies that for all $k \geq 1$, $d(M^{k+1}) > d(M^k)$ (otherwise it would also hold that $d(M^{2k}) \leq d(M^k)$, violating the non-triviality condition). In fact, similarly $d(A^{k+1}) > d(A^k)$, for any infinite definable A. Furthermore, the nontriviality also implies that $d(A) \neq 0$ for any infinite definable A.

The next proposition is a characterization of structures admitting a non-trivial dimension function.

Theorem 3.2 *M admits a non-trivial dimension function iff it satisfies the weak pigeonhole principle $WPHP^{n^2}_n$: There is no definable set A with at least two elements and a definable embedding of A^2 into A.*

Proof :

Assume that M admits a non-trivial dimension function. Hence $d(A^2) > d(A)$ for all infinite definable sets A. This inequality prevents a definable embedding of A^2 into A as such an embedding would imply $d(A^2) \leq d(A)$. As $\mathrm{WPHP}^{n^2}_n$ automatically holds for finite sets, M satisfies $\mathrm{WPHP}^{n^2}_n$.

For the opposite direction assume that M satisfies $\mathrm{WPHP}^{n^2}_n$. It is sufficient to show that $dim(A^2) > dim(A)$ in the universal dimension function $dim : \mathrm{Def}(M) \to \mathcal{D}(M)$, defined in [9]. The construction of the universal dim is simple. Define an equivalence relation $\equiv$ on $\mathrm{Def}(M)$ by: $A \equiv B$ iff for some $k, \ell \in \mathbf{N}$, A definably embeds into k copies of B and B definably embeds into ℓ disjoint copies of A. The semiring $\mathcal{D}(M)$ is $\mathrm{Def}(M)/\equiv$, and the universal dimension function d is the quotient map. To prove that $dim(A^2) > dim(A)$ we need to show, for infinite definable A, that A^2 cannot definably embed into k copies of A, any $k \in \mathbf{N}$. Assume f is such a definable embedding. Define another embedding $g : A^4 \to A^2$ by mapping first $(x, y, z, t) \in A^4$ to $(f(x, y), f(z, t)) \in (k \cdot A)^2$, and thinking of $(k \cdot A)^2$ as k^2 copies of A^2 mapping this further by f into k^3 copies of A. Using parameters for k^3 distinct elements of A, $k^3 \cdot A$ embeds into A^2.

q.e.d.

If the structure satisfies also the CC principle we can define a combinatorially more transparent dimension function with values in $(\mathbf{R}^{\geq 0}, 0, +, \leq)$.

Definition 3.3 *Let M be a structure and let $dim : Def(M) \to \mathcal{D}(M)$ be the universal dimension function. Define $\delta : Def(M) \to \mathbf{R}^{\geq 0} \cup \{-\infty\}$ by putting $\delta(\emptyset) = -\infty$ and defining $\delta(A)$ for non-empty A as the infimum of all $\frac{k}{\ell}$ such that $k, \ell \geq 1$ and $dim(A^\ell) \leq dim(M^k)$ in $\mathcal{D}(M)$.*

Note that as A is a subset of some power of M the number $\delta(A)$ is defined.

Theorem 3.4 *Let M be a structure satisfying both $WPHP_n^{n^2}$ and CC. Then δ is a non-trivial dimension function and $\delta(M^k) = k$, for all $k \geq 1$.*

Proof :

As $\delta(A)$ depends on $dim(A)$ only, δ factors through $[\ldots]$, $\delta(\{a\}) = 0$, $\delta(A) \leq \delta(B)$ if A definably embeds into B, and $\delta(A\dot{\cup}B) = max(\delta(A), \delta(B))$.

The CC principle is used in verifying the last condition.

Claim: *$\delta(A \times B) = \delta(A) + \delta(B)$, for all definable A and B.*

Assume first $dim(A^\ell) \leq dim(M^k)$ and $dim(B^v) \leq dim(M^u)$. Then $dim(A^{\ell v} \times B^{\ell v}) \leq dim(M^{kv} \times M^{\ell u})$. As $\frac{kv+\ell u}{\ell v} = \frac{k}{\ell} + \frac{u}{v}$, this shows that always $\delta(A \times B) \leq \delta(A) + \delta(B)$.

We need to prove that the inequality cannot be strict. By CC either A definably embeds in B or vice versa. Assume the former.

Consider first the case that $\delta(A) = 0$. So for any ℓ, $dim(A^\ell) \leq dim(M^{u_\ell})$ such that $\frac{u_\ell}{\ell}$ goes to 0 as ℓ grows. Now assume that $dim(B^\ell) \leq dim(M^k)$. So for all $t \geq 1$, $dim((A \times B)^{t\ell}) \leq dim(M^{tk+u_{\ell t}})$. But $\frac{tk+u_{\ell t}}{\ell t}$ goes to $\frac{k}{\ell}$ as t grows. Hence $\delta(A \times B) = \delta(B)$ and the equality is proved in this case.

Next consider the case $\delta(A) > 0$. Let α be the infimum of all $\frac{u}{v}$ such that B^v embeds in A^u. By $\delta(A) > 0$ this is well-defined. By $\mathrm{WPHP}_n^{n^2}$ (and the argument as at the end of the proof of Theorem 3.2), $\alpha \geq 1$. By CC, either B^v embeds in A^u or vice versa. So α^{-1} is the infimum of all $\frac{u}{v}$ such that A^v embeds in B^u.

We may conveniently but somewhat loosely say that "$B^{\alpha^{-1}}$ embeds in A" and "A^α embeds in B". This precisely means that for $\frac{u}{v}$ arbitrarily close to α (resp. α^{-1}), B^v embeds in A^u (resp. A^v in B^u).

Now assume that $(A \times B)^\ell$ embeds in M^k. Then, using the terminology above, $B^{(1+\alpha^{-1})\ell}$ embeds in M^k. So $\delta(B) \leq \frac{k}{\ell(1+\alpha^{-1})}$. Similarly $A^{(1+\alpha)\ell}$ embeds in M^k and $\delta(A) \leq \frac{k}{\ell(1+\alpha)}$.

But $\frac{k}{\ell(1+\alpha^{-1})} + \frac{k}{\ell(1+\alpha)} = \frac{k}{\ell}$, which shows that $\delta(A) + \delta(B) \leq \delta(A \times B)$.

Finally, $\delta(M^k) = k$ follows from the fact that $dim(M^{t+1}) > dim(M^t)$, for all $t \geq 1$.

q.e.d.

Lemma 3.5 *Assume that M is a structure satisfying principles WPHP$_n^{2n}$ and CC. Then $\mathcal{D}(M) \setminus \{-\infty\}$ satisfies the cancellation law:*

$$d_0 \oplus d = d_1 \oplus d \to d_0 = d_1$$

Proof :

Let A_0, A_1 and B be definable sets with dimension d_0, d_1 and d respectively. Assume that $d_0 \oplus d = d_1 \oplus d$ but $d_0 \neq d_1$. By CC this means that either $d_0 < d_1$ or $d_1 < d_0$; assume the former. Then, again by CC, any finite number of copies of A_0 embeds in A_1.

By $d_0 \oplus d = d_1 \oplus d$ it follows that $A_1 \times B$ embeds in k copies of $A_0 \times B$, some $k \geq 1$. On the other hand, $2k$ copies of A_0 embed in A_1, so also $2k$ copies of $A_0 \times B$ embed in $A_1 \times B$, and hence also in k copies of $A_0 \times B$. This violates the WPHP$_n^{2n}$ principle.

q.e.d.

We conclude the section with a notion of independence in partially ordered semigroups that will be useful in the next section. It stems from the fact that a partially ordered semigroup S satisfying cancellation naturally generalizes to a $\mathbf{Q}$-vector space.

Definition 3.6 *Let $(S, 0, \oplus, \leq)$ be a partially ordered semigroup with the cancellation law. A set $D \subseteq S \setminus \{0\}$ is* independent *iff for any $n \geq 1$, any $d_1, \ldots, d_n \in D$, and any $u_i, v_i \in \mathbf{N}$ for $i \leq n$ it holds:*

- *If $\sum_i u_i \cdot d_i = \sum_i v_i \cdot d_i$ then $u_i = v_i$ for all $i \leq n$.*

The following is obvious.

Lemma 3.7 *Let $(S, 0, \oplus, \leq)$ be a partially ordered semigroup with the cancellation law. Then $\{d\}$ is an independent set for any non-zero $d \in S$. Hence there is a non-empty maximal independent subset of S.*

4 Approximate Euler characteristic

In complexity theory the error in approximate counting of $A \subseteq \{0,1\}^n$ is a percentage that gets smaller as n increases. In our ideal situation with infinite ambient space the error will be infinitesimal.

Definition 4.1 *Let R be a partially ordered ring. Define three relations on R:*

1. *$a << b$ iff for all $k \in \mathbf{N}$, $k \cdot a < b$.*
2. *$a \dot{\leq} b$ iff for any rational $q > 1$ there are $k, \ell \in \mathbf{N}$ such that $\frac{\ell}{k} < q$ and $k \cdot a < \ell \cdot b$.*
3. *$a \doteq b$ iff $a \dot{\leq} b \wedge b \dot{\leq} a$.*

The general idea of axioms of aEc is to replace all original equalities $a = b$ (or inequalities $a \leq b$) in Definitions 2.1 and 2.3 by $a \doteq b$ (or by $a \dot{\leq} b$).

Definition 4.2 *A weak approximate Euler characteristic on M (aEc) is a function*

$$\xi \ : \ Def(M) \to R$$

where R is a partially ordered ring, satisfying the following inequalities for all definable sets A, B:

1. *$\xi(A) = |A|$, for finite A.*
2. *$\xi(A \dot{\cup} B) \doteq \xi(A) + \xi(B)$.*
3. *$\xi(A \times B) \doteq \xi(A) \cdot \xi(B)$.*
4. *$\xi(A) \dot{\leq} \xi(B)$, if A is definably embedded into B.*

One could define strong aEc by adding, whenever $f : A \to B$ is a definable map and $c \in R$, the condition

- $\xi(A) \doteq c \cdot \xi(B)$, if $\forall b \in B$ $\xi(f^{(-1)}(b)) \doteq c$.

However, a pair of conditions of the form:

- $\xi(A) \dot{\leq} c \cdot \xi(B)$, if $\forall b \in B$ $\xi(f^{(-1)}(b)) \dot{\leq} c$.
- $c \cdot \xi(B) \dot{\leq} \xi(A)$, if $\forall b \in B$ $c \dot{\leq} \xi(f^{(-1)}(b))$.

for $f : A \mapsto B$ is definable and injective and $c \in R$ is useful for proving inequalities of the following type. For $C \subseteq A \times B$, if $\xi(C) \leq \frac{1}{2}\xi(A \times B)$ then there is $b \in B$ such that the section $C_b := \{a \in A \mid (a, b) \in C\}$ has aEc

$\xi(C_b)\dot{\leq}\frac{1}{2}\xi(A)$. We shall not get into details about strong aEc as we do not use it anywhere in the paper.

We shall see in the proof of the next theorem that the first condition is largely cosmetic: Whenever ξ satisfies conditions 2. - 4., it can be modified on finite sets in order to satisfy the first condition too[2].

Theorem 4.3 *If a structure M admits a non-trivial weak approximate Euler characteristic then it satisfies weak pigeonhole principle $WPHP^{2n}_n$: There is no definable set $A \neq \emptyset$ and a definable embedding of two disjoint copies of A into A.*

On the other hand, if M satisfies $WPHP^{2n}_n$ and principle CC: For any two definable sets A and B, either A definably embeds in B or vice versa, then it admits a non-trivial weak approximate Euler characteristic.

Proof :

Consider the first part. Assume that ξ is a weak aEc. We first observe that $\xi(A) > 0$ for all non-empty A. As $\emptyset \subseteq A$, $\xi(\emptyset) = 0\dot{\leq}\xi(A)$ and so $0 \leq \xi(A)$. Further, a singleton embeds into A, i.e. $1\dot{\leq}\xi(A)$ and hence $0 \neq \xi(A)$. So $0 < \xi(A)$.

To prove WPHP^{2n}_n, assume for the sake of contradiction that two copies of a non-empty A embed into A. Then $\xi(A\dot{\cup}A)\dot{\leq}\xi(A)$. We have $2\xi(A)\dot{=}\xi(A\dot{\cup}A)$ and so $2\xi(A)\dot{\leq}\xi(A)$. This means that there are $k, \ell \in \mathbf{N}$ with $\frac{\ell}{k} > 1$ arbitrarily close to 1, such that $2k\xi(A) \leq \ell\xi(A)$. Taking such k, ℓ with $\frac{\ell}{k} < 2$ gives a contradiction with $0 < \xi(A)$: $0 < \xi(A)$ implies $0 < (2k - \ell)\xi(A)$ and hence $\ell\xi(A) < 2k\xi(A)$.

The proof of the second part of the theorem is divided into several claims. First note that WPHP^{2n}_n implies $\mathrm{WPHP}^{n^2}_n$ and so M does admit (by Section 3) nontrivial dimension function.

Recall the universal dimension function dim on $\mathrm{Def}(M)$ from the proof of Theorem 3.2. As dim factors through the quotient map $[\ldots] : \mathrm{Def}(M) \rightarrow \widetilde{\mathrm{Def}(M)}$, we shall sometimes write $dim(a) = d$ instead of $dim(A) = d$ for some A with $[A] = a$. Then dim is the quotient map given by the equivalence relation

$$\exists k, \ell \geq 1; a \leq k \cdot b \wedge b \leq \ell \cdot a$$

where $\leq$ is the partial ordering in $\widetilde{\mathrm{Def}(M)}$ induced by the embedability in $\mathrm{Def}(M)$. By CC the ordering is, in fact, linear. We shall denote by

[2] In fact, as pointed out by the referee, for reasons of complexity of ξ (or when applying the construction to incomplete theories) it could be best to simply remove condition 1. altogether.

$(S, 0, \oplus, \leq)$ the linearly ordered semigroup obtained from $\mathcal{D}(M)$ by deleting $-\infty$.

Fix $d \in S$; i.e. d is the dimension of a non-empty set. Let $E_d \subseteq \widetilde{\mathrm{Def}(M)}$ be the set of all $a \in \widetilde{\mathrm{Def}(M)}$ of dimension d, together with 0. Clearly, $(E_d, 0, +)$ is a sub-semigroup of the additive semigroup of $\widetilde{\mathrm{Def}(M)}$.

Claim 1: $(E_d, 0, +)$ homomorphically maps onto a sub-semigroup of the additive semigroup of the non-negative reals $(\mathbf{R}^{\geq 0}, 0, +)$, such that only 0 maps to 0.

To prove the claim fix an element $e \in E_d$. For any non-zero $a \in E_d$ define the set of rational numbers

$$C_e(a) := \{\frac{\ell}{k} \mid k \cdot a \leq \ell \cdot e, \text{ both } k, l \geq 1\}$$

As the dimensions of a and e are the same, there is some ℓ such that $a \leq \ell \cdot e$ and so $C_e(a)$ is non-empty. Define the real number $c_e(a) \geq 0$ to be the infimum of $C_e(a)$.

First note that actually $c_e(a) > 0$: We have $e \leq m \cdot a$ for some $m \geq 1$, and so $k \cdot a \leq \ell \cdot e$ implies $k \cdot a \leq (\ell m) \cdot a$. If it were that $c_e(a) = 0$ we could find $k, \ell \geq 1$ such that $\frac{\ell}{k} < \frac{1}{2m}$, i.e. $2\ell m < k$ and the inequality $k \cdot a \leq (\ell m) \cdot a$ would show that WPHP^{2n}_n fails in M for a set A for which $[A] = (\ell m) \cdot a$.

Further define $c_e(0) := 0$. The map

$$c_e : (E_d, 0, +) \to (\mathbf{R}^{\geq 0}, 0, +)$$

is a homomorphism: If $k \cdot a \leq \ell \cdot e$ and $k' \cdot a' \leq \ell' \cdot e$, then $(kk') \cdot (a + a') \leq (k'\ell + \ell' k) \cdot e$. So $C_e(a) + C_e(a') \subseteq C_e(a + a')$.

But similarly one shows that $C_e(a) + C_e(a')$ is downward cofinal in $C_e(a+a')$ (this part is analogous to the proof of the claim in the proof of Theorem 3.4). Assume $k \cdot (a + a') \leq \ell \cdot e$. We need to demonstrate that $u \cdot a \leq v \cdot e$, $u' \cdot a' \leq v' \cdot e$ for some $u, v, u', v' \geq 1$ such that $\frac{v}{u} + \frac{v'}{u'}$ is arbitrarily close to $\frac{\ell}{k}$. To simplify the notation let us write $a \leq c_{a'}(a) \cdot a'$ (meaning that we can find $m, n \geq 1$ with $\frac{n}{m}$ arbitrarily close to $c_{a'}(a)$ such that $m \cdot a \leq n \cdot a'$) and similarly $a' \leq c_a(a') \cdot a$. Then the assumption $k \cdot (a + a') \leq \ell \cdot e$ implies $k(1 + c_{a'}(a)) \cdot a' \leq \ell \cdot e$ (in the proper formulation with m, n this is $k(m + n) \cdot a' \leq \ell m \cdot e$ obtained from $k \cdot (a + a') \leq \ell \cdot e$ by first multiplying both sides by m and then replacing $m \cdot a$ by $n \cdot a'$) and $k(1 + c_a(a')) \cdot a \leq \ell \cdot e$. So it is enough to show that

$$\frac{\ell}{k(1 + c_{a'}(a))} + \frac{\ell}{k(1 + c_a(a'))} = \frac{\ell}{k}$$

WPHP^{2n}_n implies $c_a(a) = 1$ and as $c_a(a') \cdot c_{a'}(a) \geq c_a(a)$ we have $c_a(a') \cdot c_{a'}(a) \geq 1$. That the inequality cannot be strict follows by CC: Either $m \cdot a \leq n \cdot a'$ or $n \cdot a' \leq m \cdot a$, i.e. either $\frac{n}{m} \in C_{a'}(a)$ or $\frac{m}{n} \in C_a(a')$. This proves the claim.

Note that the same argument works equally well in a more general situation (whose special case is Claim 1).

Claim 2: $c_{[A]}([B]) \cdot c_{[A']}([B']) = c_{[A \times A']}([B \times B'])$, whenever both sides are defined.

Claim 3: $\mathcal{D}(M)$ is linearly ordered. Further, if $dim(A) < dim(B)$, then k copies of A embed in B, for any $k \geq 1$.

$\mathcal{D}(M)$ is clearly linearly ordered. Assume that k copies of A do not embed in B. Then, by CC, B embeds into k copies of A and so $dim(B) \leq dim(A)$ which contradicts the assumption that $dim(A) < dim(B)$.

Note that the construction of Claim 1 also yields the following characterization of the relation $c_a(b) = 1$.

Claim 4: For $a, b \in E_d$ define $a \equiv b$ iff there are definable sets A and B with $[A] = a$ and $[B] = b$ and $dim(A \triangle B) < d$ ($A \triangle B$ is the symmetric difference). Then $a \equiv b$ iff $c_a(b) = 1$.

Denote by $[A]^*$ the $\equiv$-class of $[A]$. By Claims 1 and 4, $[A]^*$ maps additive semigroup $(E_d, 0, +)$ into $(\mathbf{R}^{\geq 0}, 0, +)$. So we want to define the aEc as $[A]^*$ taking the value in the direct sum of copies of $(\mathbf{R}^{\geq 0}, 0, +)$, graded by $(S, 0, \oplus, \leq)$, i.e. in $\mathbf{R}[x^S]$. A homomorphism of $(E_d, +)$ into $(\mathbf{R}^{>0}, +)$ is a positive multiple of $c_a(x)$, any $a \in E_d$ by Claim 1, and hence we define the aEc in the following way.

Let $\{e_d\}_d$ be a fixed set of points, one from each E_d for each dimension $d \neq -\infty$. Let function $\mathcal{E} : d \in S \to \epsilon_d \in \mathbf{R}^{>0}$ be arbitrary. Define map

$$\xi_{\mathcal{E}} : \mathrm{Def}(M) \to \mathbf{R}[x^S]$$

for a non-empty definable A by

$$\xi_{\mathcal{E}}(A) := \epsilon_d c_{e_d}([A]) \cdot x^{dim(A)}$$

We will show in Claims 5 and 6 that one can choose points $\{e_d\}_d$ and function $\mathcal{E}$ so that $\xi_{\mathcal{E}}$ is the desired aEc.

Intuitively, one would like to have e_d's the "unit d-dimensional cubes" in which case all ϵ_d's could be equal to 1. However, in a general situation there is no such object and so we pick e_d's arbitrarily and use ϵ_d's for a rectification of the values, so that they obey conditions 1. and 3. of Definition 4.2.

Claim 5: Assume e_0 is the $[\ldots]$-class of a finite set with $n \geq 1$ elements and $\epsilon_0 = n$. Then for any finite A: $\xi_{\mathcal{E}}(A) = |A|$.

The value of $\xi_{\mathcal{E}}(A)$ is n times the infimum of all $\frac{\ell}{k}$ such that k copies of A embed into an ℓn-element set. The infimum is clearly $\frac{|A|}{n}$.

The next claim will ultimately follow from Claim 2 and could be proved directly if we could use an induction on the dimension. However, we need to proceed more generally in order to apply to any possible semigroup S where an induction argument is not available.

Claim 6: For any ϵ there is a set $\{e_d\}_{d \neq -\infty}$ with $e_d \in E_d$ and function $\mathcal{E}$ with $\epsilon_0 = \epsilon$ such that the resulting function $\xi_{\mathcal{E}}$ satisfies the multiplicative condition 3. of Definition 4.2.

By Lemmas 3.5 and 3.7 semigroup S satisfies the cancellation law and there is a non-empty maximal independent (in the sense of Definition 3.6) subset $D \subseteq S \setminus \{0\}$. Let A_d, $d \in S \setminus \{0\}$, be some fixed sets such that $[A_d] = e_d \in E_d$.

Define ϵ_d, for $d \neq 0$, as follows. If $d \in D$, $\epsilon_d := 1$. If $d \notin D$ then, by the maximality of D, there are n, d_i and u_i, v_i for $i \leq n$, and $u \geq 1$ such that

$$\sum_i u_i \cdot d_i + u \cdot d = \sum_i v_i \cdot d_i$$

(by the cancellation law we may assume that d appears only on one side of the equation; similarly we could assume that $u_i v_i = 0$ for all $i \leq n$). Then define:

$$\epsilon_d \ := \ c_{[\Pi_i A_{d_i}^{v_i}]}([\Pi_i A_{d_i}^{u_i} \times A_d^u])^{1/u}$$

where Π_i is a Cartesian product. The intuition behind this definition is this: A_d's for $d \in D$ are postulated to have the "unit size". The "right size" of A_d for $d \notin D$ is then computed using a relation between Cartesian products involving the set and some unit size sets, thinking that the Cartesian u-th power increases the size to exponent u.

By the equality $\sum_i u_i \cdot d_i + u \cdot d = \sum_i v_i \cdot d_i$ the sets $\Pi_i A_{d_i}^{v_i}$ and $\Pi_i A_{d_i}^{u_i} \times A_d^u$ have the same dimension and so the term defining ϵ_d is well-defined. By the usual manipulations with linear equalities (this we can do having the cancellation law) there is, for any given d, the minimal u for which an equality like $\sum_i u_i \cdot d_i + u \cdot d = \sum_i v_i \cdot d_i$ holds and all other equalities of this type are its integer multiple. This shows that the value of ϵ_d is independent of the choice of the particular equation and u. Here we use the property $c_{[A]}([B]) \cdot c_{[A']}([B']) = c_{[A\times A']}([B \times B'])$ from Claim 2 (and we shall use it repeatedly without further mentioning).

To demonstrate the multiplicative property $\xi_{\mathcal{E}}(U) \cdot \xi_{\mathcal{E}}(V) = \xi_{\mathcal{E}}(U \times V)$ assume that $dim(U) = d_0$ and $dim(V) = d_1$. Then we want to show:

$$\epsilon_{d_0} c_{[A_{d_0}]}([U]) \epsilon_{d_1} c_{[A_{d_1}]}([V]) = \epsilon_{d_0 \oplus d_1} c_{[A_{d_0 \oplus d_1}]}([U \times V])$$

As

$$c_{[A_{d_0}]}([U]) c_{[A_{d_1}]}([V]) = c_{[A_{d_0} \times A_{d_1}]}([U \times V]) = c_{[A_{d_0} \times A_{d_1}]}([A_{d_0 \oplus d_1}]) c_{[A_{d_0 \oplus d_1}]}([U \times V])$$

and

$$c_{[A_{d_0} \times A_{d_1}]}([A_{d_0 \oplus d_1}])^{-1} = c_{[A_{d_0 \oplus d_1}]}([A_{d_0} \times A_{d_1}])$$

it is enough to show that:

$$\epsilon_{d_0} \epsilon_{d_1} = \epsilon_{d_0 \oplus d_1} c_{[A_{d_0 \oplus d_1}]}([A_{d_0} \times A_{d_1}])$$

The values of the ϵ' are computed from some linear dependence relations in S involving integer scalar multiples of d_0, d_1 and $d_0 \oplus d_1$ respectively. By taking some common multiple we may assume that the dimensions appear in the liner dependencies with scalar $u \geq 1$. In particular,

$$\epsilon_{d_0} = c_{[B_0]}([B_1 \times A^u_{d_0}])^{1/u} \quad \text{and} \quad \epsilon_{d_1} = c_{[C_0]}([C_1 \times A^u_{d_1}])^{1/u}$$

where B_0, B_1, C_0, C_1 are Cartesian products of A_d's for some $d \in D$ (i.e. of the sets we postulated to have unit size, so we may think of B_0, B_1, C_0, C_1 as "unit cubes"). It holds:

$$c_{[B_0]}([B_1 \times A^u_{d_0}])^{1/u} c_{[C_0]}([C_1 \times A^u_{d_1}])^{1/u} = c_{[B_0 \times C_0]}([B_1 \times C_1 \times (A_{d_0} \times A_{d_1})^u])^{1/u} =$$

$$c_{[B_0 \times C_0]}([B_1 \times C_1 \times A^u_{d_0 \oplus d_1}])^{1/u} c_{[A^u_{d_0 \oplus d_1}]}([(A_{d_0} \times A_{d_1})^u])^{1/u}$$

The left-hand side is $\epsilon_{d_0} \epsilon_{d_1}$, while the right-hand side is $\epsilon_{d_0 \oplus d_1} c_{[A_{d_0 \oplus d_1}]}([A_{d_0} \times A_{d_1}])$. This proves the claim.

We are ready now to prove the second part of the theorem. Take $\mathcal{E}$ provided by Claim 6, with ϵ_0 chosen so as to satisfy the hypothesis of Claim 5. Hence the map $\xi_{\mathcal{E}}$ satisfies conditions 1. and 3. of Definition 4.2, by Claims 5 and 6.

Condition 4. is satisfied for any $\mathcal{E}$. Assume A embeds into B. Then either $dim(A) < dim(B)$ or $dim(A) = dim(B)$. In the former case, by Claim 3, even $dim(A) << dim(B)$. In the latter case $c_{[B]}([A]) \leq 1$, so $c_e([A]) \leq c_e([B])$ for $e = e_{dim(A)}$, and so $\xi_{\mathcal{E}}(A) \leq \xi_{\mathcal{E}}(B)$.

It remains to verify condition 2.: Let A, B be two disjoint definable sets and assume first that $dim(A) < dim(B)$. Then $\xi_{\mathcal{E}}(A \cup B) = \xi_{\mathcal{E}}(B)$ and $\xi_{\mathcal{E}}(A) << \xi_{\mathcal{E}}(B)$, so $\xi_{\mathcal{E}}(A \cup B) \doteq \xi_{\mathcal{E}}(A) + \xi_{\mathcal{E}}(B)$ (as $x^{dim(A)} << x^{dim(B)}$).

Next assume $dim(A) = dim(B)$. Then even $\xi_{\mathcal{E}}(A \cup B) = \xi_{\mathcal{E}}(A) + \xi_{\mathcal{E}}(B)$ by Claim 1.

q.e.d.

It would be interesting to avoid using principle CC in the theorem. This could be done if, for example, every structure satisfying WPHP_n^{2n} had an expansion still satisfying WPHP_n^{2n} but also satisfying CC. Is this true?

Also, when could the role of $(\mathbf{R}^{\geq 0}, 0, +, \leq)$ (both as coefficients or as degrees) in $\mathbf{R}[x^S]$ be taken by $(\mathbf{Q}^{\geq 0}, 0, +, \leq)$?

5 Examples

We conclude the paper by few examples of well-known structures to illustrate some facts and notions discussed in the paper.

1. Example: reals R.

The real closed field $\mathbf{R}$ satisfies the ontoPHP (as bijections need to preserve Ec and Ec of a set and of the set plus one point differ; $\mathbf{R}$ admits Ec - see e.g. [4]) but clearly not PHP or WPHP_n^{2n}. Hence $\mathbf{R}$ admits neither ordered Ec nor aEc. On the other hand $\mathbf{R}$ admits a dimension function (see e.g. [4]) and satisfies $\mathrm{WPHP}_n^{n^2}$ (by dimension reasons). $\mathbf{R}$ also satisfies CC, cf. [9].

2. Example: complex numbers C.

Complex numbers satisfy PHP (by a theorem of Ax [2]) and hence the other three (W)PHP principles too. The universal Ec is thus ordered and so it is also aEc. The Grothendieck ring $K_0(\mathbf{C})$ is huge; for example, it contains the ring of complex polynomials in continuum many unknowns (cf. [9]). Principle CC fails for $\mathbf{C}$.

3. Example: structures with global ranks.

Let $d : \mathrm{Def}(M) \to S$ be a dimension function with $(S, -\infty, 0, \vee, \leq, \oplus)$ linearly ordered. Define

$$deg_d \ : \ \mathrm{Def}(M) \to \mathbf{N} \cup \{\infty\}$$

as the maximum $n \in \mathbf{N}$ s.t. there are n disjoint subsets of A each of the same d-dimension as A, and $deg_d(A) := \infty$, if no such n exists.

A good example of such d is Morley rank RM in almost strongly minimal structures[3]; values of deg_{RM} are in $\mathbf{N}$, for example, for strongly minimal structures. As a simple specific example we take the structure $M := (\mathbf{N}, suc)$. As noted in [9, Sec.8], M satisfies WPHPs while obviously not ontoPHP. Definable subsets $A \subseteq M^n$ are disjoint unions of sets A_i of the form

$$U_1 \times \ldots \times U_n$$

where each $U_i \subseteq \mathbf{N}$ is either finite or co-finite. Define $d(A)$ to be the maximum number of co-finite in some A_i. deg_d has values in $\mathbf{N}$.

Define function $\xi \ : \ \mathrm{Def}(M) \to \mathbf{N}[x^S]$ by

$$\xi(A) := deg_d(A) \cdot x^{d(A)}$$

Then ξ is an aEc.

To see this note that for A finite, $d(A) = 0$ and $deg_d(A) = |A|$. So $\xi(A) = |A| \cdot x^0 = |A|$.

For the additivity conditions let $d(A) = u \leq v = d(B)$. If $u < v$, $deg_d(A \dot\cup B) = deg_d(B)$, so

$$\xi(A \dot\cup B) = deg_d(A \dot\cup B) x^v = \xi(B) \leq \xi(A) + \xi(B) \ .$$

If $u = v$, $deg_d(A \dot\cup B) = deg_d(A) + deg_d(B)$, and

$$\xi(A \dot\cup B) = (deg_d(A) + deg_d(B)) x^v = \xi(A) + \xi(B) \ .$$

For the opposite inequality $\xi(A) + \xi(B) \dot\leq \xi(A \dot\cup B)$ write

$$\xi(A) + \xi(B) = deg_d(A) x^u \ + \ deg_d(B) x^v$$

If $u < v$, $deg_d(A) x^u << x^v$, so $\xi(A) + \xi(B) \dot\leq \xi(B) \leq \xi(A \dot\cup B)$. If $u = v$, $\xi(A) + \xi(B) = \xi(A \dot\cup B)$.

[3] As pointed out by the referee, more generally one can take Lascar rank in structures of finite Lascar rank.

I leave it to the reader to check the other two properties required from aEc.

4. Example: pseudo-finite fields.

Let F be an ultraproduct of finite fields $\mathbf{F}_q$'s. Chatzidakis, van den Dries and Macintyre [3] assign to all definable A in F a pair (μ, d), with μ a positive rational and d a natural number such that for some X in the ultrafilter and all $q \in X$ it holds:

$$|A(\mathbf{F}_q) - \mu \cdot q^d| = O(q^{d-\frac{1}{2}})$$

This also yields an aEc

$$A \longrightarrow \mu \cdot x^d \in \mathbf{Q}[x]$$

(changing values for finite A to $|A|$).

5. Example: bounded arithmetic.

This is another example of a structure satisfying WPHP_n^{2n} but not ontoPHP. Let $(M, 0, 1, \oplus, \odot)$ be a countable non-standard model of true arithmetic with ternary relations $\oplus$ and $\odot$ for graphs of additions and multiplication instead of the functions themselves. Let $n \in M$ be a non-standard number and M_n the substructure with the universe $[0, \ldots, n]$.

One can find a bijection f (not definable in M_n) between $[0, \ldots, n]$ and $[0, \ldots, n-1]$ (and so enforcing a failure of ontoPHP) such that the expanded structure still satisfies WPHP_n^{2n} (and also induction for all formulas, possibly containing symbol f); this follows from [10, 8, 11], see also [9].

It is an interesting open problem in bounded arithmetic whether we can similarly violate by f one of the weak pigeonhole principles, but in a way so that induction for all formulas is maintained. This has close relations to propositional proof complexity too.

Let us close by a general remark the referee made and which I consider important. The constructions in the paper do not really use the fact that we operate with a semiring derived from a first-order structure. Semiring $\widetilde{\mathrm{Def}(M)}$ is more here important than the class of definable sets $\mathrm{Def}(M)$. Combinatorial properties of structures are reflected in first-order properties of $\widetilde{\mathrm{Def}(M)}$ (for example, WPHP_n^{2n} implies that $\widetilde{\mathrm{Def}(M)}$ satisfies $\forall x, y, z; x + x + y + z \neq x + z$ and CC implies the validity of $\forall x, y \exists z; (x + z = y \vee y + z = x))$ and that is the only way the principles are used in the constructions. Thus it could make a good sense to work purely abstractly with semirings. Furthermore, this abstract approach allows to split the question about the importance of CC (raised at the end of Section 4) into two separate questions. First, can one find a commutative semiring satisfying WPHP_n^{2n} but not admitting nontrivial weak aEc? And secondly, can such a semiring take the form of $\widetilde{\mathrm{Def}(M)}$ for some first order structure M?

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