Problem 4 and some remarks

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Problem 4 solution. We will work in arbitrary $M \models I\Delta_0$ and proceed by contradiction. We assume there exists $x \in M$ and Δ_0-definable $f : [x] \to [|x|-1]$ such that f is injective, where $[x] = \{0, \dots, x-1\}$.

Let $A_f(x, y)$ be the formula defining f. Then we define

$$\varphi(x) = (\exists y < x)(A(y, x) \wedge x \notin y). \tag{1}$$

Since $A_f(x, y)$ is Δ_0 then so is φ.

From `notes-26-3-20.pdf` we have $I\Delta_0 \vdash \mathrm{CA}$. So we know, that there exists $s < x$ such that for all $z \in M$

$$z \in s \Leftrightarrow \varphi(z). \tag{2}$$

Now is $f(s) \in s$? If $f(s) \notin s$ then definitely $\varphi(f(s))$ so by (2) we have $f(s) \in s$ which is a contradiction.

So we have $f(s) \in s$ and again by (2) we have that there exists $y < x$ such that $f(s) = f(y)$ and $f(y) \notin y$ therefore $f(x) \neq f(y)$ so f cannot be injective. A contradiction. □

Remarks and questions.

1. We got inspired for this argument thanks to the hint from prof. Krajíček. The argument here is actually *dual* to Cantor's diagonal argument.

   In Cantor's diagonal argument we assume $f : X \to \mathcal{P}(X)$ to be onto and show the set $\{x \in X; x \notin f(x)\}$ cannot be in the range of f. In set theory we can easily reverse arrows and get the version used above. However this is equivalent to the axiom of choice since we have to pick one of the elements in the preimage of every $x \in X$ to construct an injective function from X to $\mathcal{P}(x)$.

   So our question is what about the dual version of the pigeonhole principle? Is the statement that there's no Δ_0-definable surjection from $[x]$ to $[x+1]$ equivalent to PHP?

2. We also considered proving $WPHP^x_{|x|-1}$ directly by Δ_0-induction. We noticed that if we could prove that for all x we have

   $$(|x+1| - 1) - (|x| - 1) \leq 1 \tag{$*$}$$

we could prove the induction step. That is we would just construct from injection $f : [x + 1] \to [|x + 1| - 1]$ different injection $f : [x] \to [|x| - 1]$ by swapping the preimage of $|x| - 1$ with x. We did not try to prove $(*)$ since we were not sure what were the axioms defining $|x|$ or whether it is simply a shorthand for the formula defining the graph of exponential.

3. We looked briefly at **problems 2** and **3**. We believe that under the assumption of CA the construction from `notes-26-3-20.pdf` is not that difficult to prove to be well-defined. We did not get to write out any details.

   However we have an idea for extending the construction to $WPHP_x^{x^2}$. First we close J under multiplication and call it J^*. And define the relation $Q^*(x, z)$ as

$$P(x, y) \wedge ((x \in J^* \wedge y = z) \vee (y \in J^* \wedge y = z \cdot z)). \tag{3}$$

   J^* being closed under multiplication seems to be enough for this to work.