Hyperbolické funkce

sinh, cosh

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \ldots \quad \text{lichá fce}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \ldots \quad \text{sudá fce}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

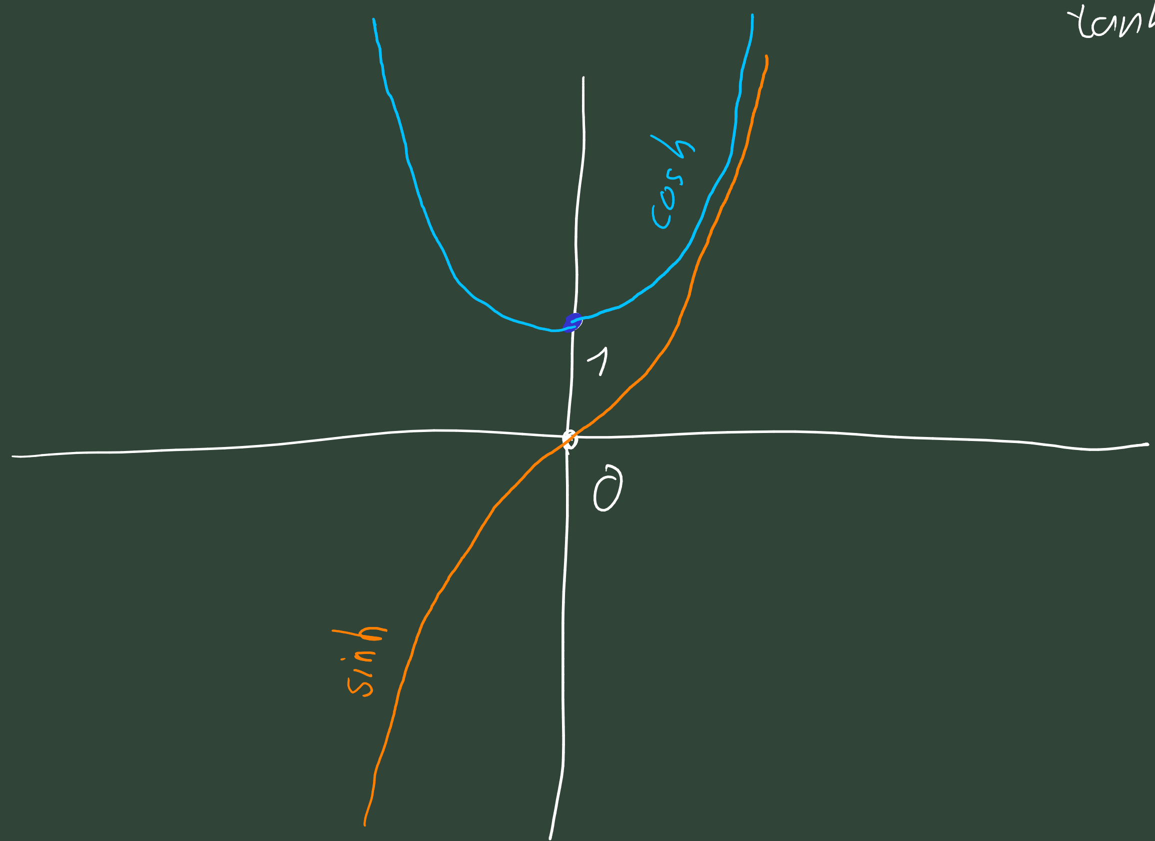
$$(\sin x)' = \cos x$$

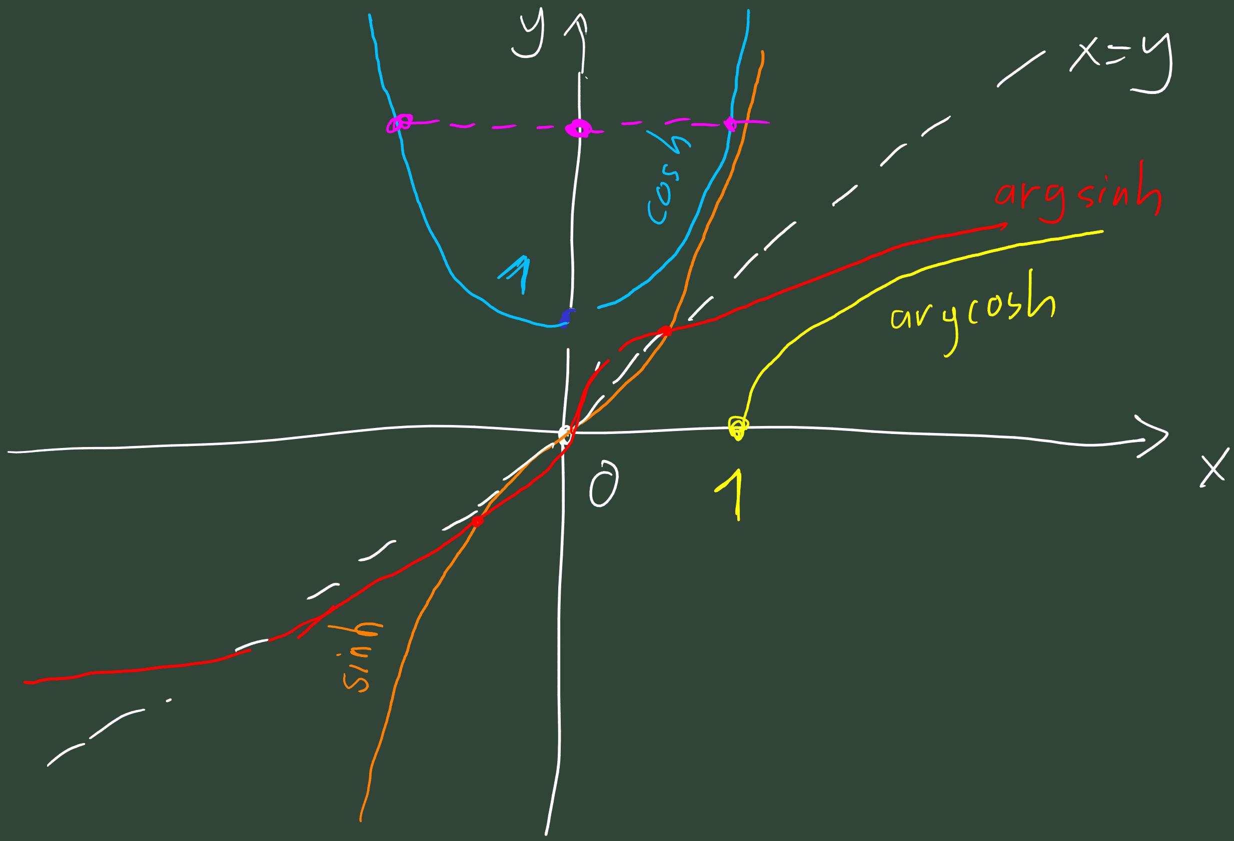
$$(\cos x)' = -\sin x$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$\tanh = \frac{\sinh}{\cosh}$$



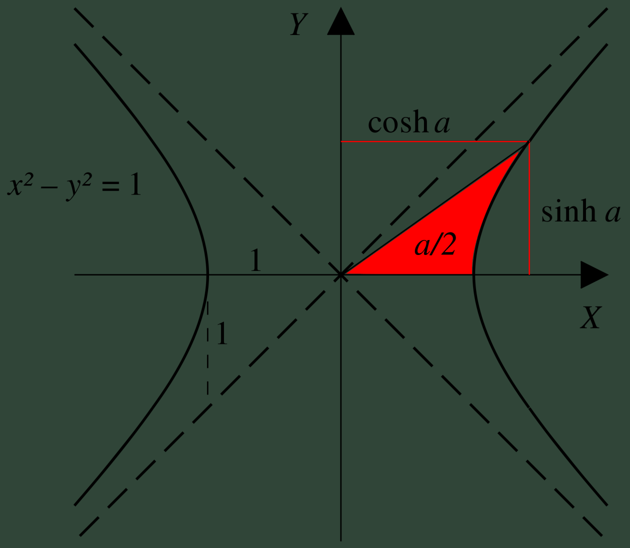


$$\operatorname{argsinh} = (\sinh)^{-1} : \mathbb{R} \to \mathbb{R}$$

$$= \operatorname{arsinh} \quad (\text{"area sinus hyperbolicus"})$$

$$= \operatorname{asinh}$$

$$= \cancel{\operatorname{arc\,sinh}}$$



$$\operatorname{argcosh} = \left(\cosh\big|_{[0,+\infty)}\right)^{-1}$$

$$e^x = \sinh x + \cosh x$$

$$\Updownarrow$$

$$\boxed{e^{ix} = i\sin x + \cos x}$$

$$\boxed{e^x = \sinh x + \cosh x}$$

$$1 = \cosh^2 x - \sinh^2 x$$

$$\cosh x = \sqrt{1 + \sinh^2 x}, \quad x \in \mathbb{R}$$

$$\sinh x = \sqrt{\cosh^2 x - 1}, \quad x \geq 0$$

$$\Rightarrow \quad e^x = \sinh x + \sqrt{1 + \sinh^2 x}, \quad x \in \mathbb{R}$$

$$x = \ln\left(\sinh x + \sqrt{1 + \sinh^2 x}\right), \quad x \in \mathbb{R}$$

$$\boxed{\operatorname{argsinh} y = \ln\left(y + \sqrt{1 + y^2}\right)}, \quad y \in \mathbb{R}$$

$$e^x = \sqrt{\cosh^2 x - 1} + \cosh x \ , \ x \geq 0$$

$$x = \ln\left(\cosh x + \sqrt{\cosh^2 x - 1}\right), \ x \geq 0$$

$$\boxed{\operatorname{argcosh} y = \ln\left(y + \sqrt{y^2 - 1}\right), \ y \geq 1}$$

$$\left(\operatorname{argsinh} y\right)' = \frac{1}{\sqrt{1+y^2}} \ , \ y \in \mathbb{R}$$

$$\sinh' x = \cosh x = \sqrt{1 + \sinh^2 x}$$

$$\left(\operatorname{argcosh} y\right)' = \frac{1}{\sqrt{y^2 - 1}} \ , \ y \geq 1$$

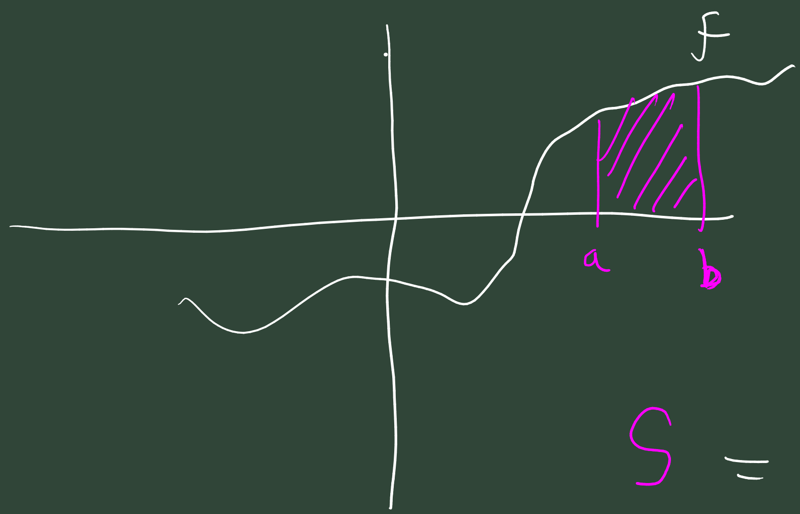
Neurčitý integrál (primitivní funkce)

Úkol: Nepřítel (učitel) nám zadá funkci

$f : I \longrightarrow \mathbb{R}$, I je otevřený interval

Chceme najít $F : I \longrightarrow \mathbb{R}$

tak, aby $F'(x) = f(x)$, $\forall x \in I$



$S = F(b) - F(a)$

F = primitivní funkce

$$\tilde{F} = F + C, \quad C \in \mathbb{R}$$

$$(\tilde{F})' = F' + 0 = f$$

primitivní fce jsou vždy určeny až na aditivní konstanta C

$$\int f(x)\,dx = F + C \quad (\text{na } I)$$

Př. 6)

$$\int \underbrace{x e^{-x^2}}_{f} dx = -\frac{e^{-x^2}}{2} + C$$

$$(e^{-x^2})' = e^{-x^2}(-2x) \quad \Big| \cdot \left(-\frac{1}{2}\right)$$

1.

$$\int \overbrace{\left(\frac{1-x}{x}\right)^2}^{f} dx = C - \frac{1}{x} - 2\ln|x| + x$$

$$\|$$

$$\int \frac{1-2x+x^2}{x^2}\,dx = \int \frac{1}{x^2}\,dx - \int \frac{2}{x}\,dx + \int 1\,dx$$

$$\boxed{\begin{array}{ll} (x^n)' = n x^{n-1}, & \forall n \\ & (\text{když } n<1 \\ & \quad x \neq 0) \\ \int x^{n-1}\,dx = \frac{x^n}{n} + C & (\ln|x|)' = \frac{1}{x} \\ \int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \ n \neq -1 & \end{array}}$$

$$= \frac{x^{-1}}{-1} - 2\ln|x| + x + C$$

$$= -\frac{1}{x} - 2\ln|x| + x + C$$

na $(0,+\infty)$ a na $(-\infty,0)$.

2. $$\int \frac{dx}{x^2-x+2} = \int \frac{1}{x^2-x+2}\,dx$$

$$x^2-x+2 = \left(x-\frac{1}{2}\right)^2 + \frac{7}{4} > 0$$

integrand je spojitá na $\mathbb{R}$

$\rightarrow$ PF bude existovat na $\mathbb{R}$

$$\int \frac{dx}{x^2-x+2} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} \cdot \frac{\frac{4}{7}}{\frac{4}{7}} =$$

$$\boxed{\left(\arctan x\right)' = \frac{1}{1+x^2}}$$

$$= \frac{4}{7} \int \frac{dx}{\frac{4}{7}\left(x-\frac{1}{2}\right)^2+1}$$

$$= \frac{4}{7} \int \frac{dx}{\left(\frac{2}{\sqrt{7}}x-\frac{1}{\sqrt{7}}\right)^2+1} = \frac{\sqrt{7}}{2}\frac{4}{7}\operatorname{arctan}\left(\frac{2}{\sqrt{7}}x-\frac{1}{\sqrt{7}}\right) + C$$

na $\mathbb{R}$

$$F = \frac{4}{7} \operatorname{artan}\left(\frac{2}{\sqrt{7}} x - \frac{1}{\sqrt{7}} \right)$$

$$F' = \frac{4}{7} \frac{1}{\left(\frac{2}{\sqrt{7}} x - \frac{1}{\sqrt{7}} \right)^2 + 1} \cdot \frac{2}{\sqrt{7}}$$

$$G = \frac{\sqrt{7}}{2} F$$

Prestávka do 14:30

Per partes

$$\int u v' \, dx = uv - \int u' v \, dx$$

Př.

$$\int \ln x \, dx \overset{C}{=} {}^{(0,+\infty)} \int 1 \cdot \ln x \, dx =$$

$$= \left| \begin{array}{ll} \multicolumn{2}{c}{\text{per partes}} \\ u = \ln x \, , & u' = \frac{1}{x} \\ v' = 1 \, , & v = x \end{array} \right| = x \ln x -$$

$$- \int \underbrace{x \cdot \frac{1}{x}}_{1} \, dx =$$

$$= x \ln x - x + C$$

na $(0, +\infty)$

<u>ZK:</u>

$$F(x) = x \ln x - x$$

$$F'(x) = \ln x + x \cdot \cancel{\frac{1}{x}} - \cancel{1}$$

$$= \ln x$$

$$\underbrace{\int e^{3x} \cos 2x \, dx}_{I} = \left| \begin{array}{l} \text{per partes} \\ u = \cos 2x, \quad u' = -2\sin 2x \\ v' = e^{3x}, \quad v = \frac{1}{3}e^{3x} \end{array} \right.$$

$$= \overset{uv}{\overbrace{\frac{1}{3} e^{3x} \cos 2x}} - \int \overset{u'v}{\overbrace{(-2\sin 2x)\frac{1}{3}e^{3x}}} dx$$

$$= \frac{1}{3} e^{3x} \cos 2x + \frac{2}{3}\int e^{3x} \sin 2x \, dx =$$

$$= \left| \begin{array}{l} \text{per partes} \\ u = \sin 2x, \quad u' = 2\cos 2x \\ v' = e^{3x}, \quad v = \frac{1}{3}e^{3x} \end{array} \right| =$$

$$= \frac{1}{3} e^{3x} \cos 2x + \frac{1}{3} \cdot \frac{2}{3} e^{3x} \sin 2x -$$

$$- \frac{2}{3} \int 2\cos(2x) \, \frac{1}{3} e^{3x} \, dx$$

$$I = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{9} e^{3x} \sin 2x - \frac{4}{9} I$$

$$\underbrace{\left(1 + \frac{4}{9}\right)}_{\frac{13}{9}} I = \frac{1}{3} e^{3x} \cos 2x + \frac{2}{9} e^{3x} \sin 2x + C$$

$$I = \frac{3}{13} e^{3x} \cos 2x + \frac{2}{13} e^{3x} \sin 2x + C$$

$$I = \underbrace{\frac{3}{13} e^{3x} \cos 2x + \frac{2}{13} e^{3x} \sin 2x}_{F(x)} + C \quad \text{na } \mathbb{R}$$

ZK:

$$F'(x) = \frac{9}{13} e^{3x} \cos 2x - \cancel{\frac{3 \cdot 2}{13} e^{3x} \sin 2x}$$

$$+ \cancel{\frac{6}{13} e^{3x} \sin 2x} + \frac{4}{13} e^{3x} \cos 2x$$

$$= e^{3x} \cos 2x$$

1. věta o substituci

I_1, I_2 ot. intervaly

Má-li: $\varphi : I_1 \to I_2$ derivaci $\forall x \in I_1$

Pak

$$\int f(\varphi(x))\,\varphi'(x)\,dx = \int f(t)\,dt \Big|_{t=\varphi(x)}$$

(substituce $\varphi(x) = t$) (na I_1)

2. věta o substituci

$\varphi : I_1 \xrightarrow{na} I_2$ má nenulovou derivaci $\forall t \in I_1$

Pak

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt \Big|_{t=\varphi^{-1}(x)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

<u>Pr.</u>

$$\int \frac{dx}{e^x + e^{-x}} = \frac{1}{2}\int \frac{dx}{\cosh x} = \frac{1}{2}\int \frac{\cosh x\,dx}{\cosh^2 x} =$$

$$= \frac{1}{2}\int \frac{\cosh x}{1+\sinh^2 x}\,dx = \left| \begin{array}{c} \text{sub. } (1) \\ \sinh x = t \\ \cosh x\,dx = 1\,dt \end{array} \right| =$$

$$= \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\arctan t + C \quad (\text{na } \mathbb{R})$$

$$= \frac{1}{2}\arctan(\sinh x) + C$$

$$\int \frac{\ln^2 x}{x}\,dx = \frac{\ln^3 x}{3} + C$$

na $(0, +\infty)$

$$= \left| \begin{array}{l} \text{sub.} \quad (1) \\ t = \ln x \\ dt = \frac{dx}{x} \\ t \in \mathbb{R} \\ x \in (0, +\infty) \end{array} \right| = \int t^2\,dt =$$

$$= \frac{t^3}{3} + C = \frac{\ln^3 x}{3} + C$$

<u>ZK:</u>

$$F(x) = \frac{\ln^3 x}{3}$$

$$F'(x) = \frac{\cancel{3}\ln^2 x}{\cancel{3}} \cdot \frac{1}{x} \quad \checkmark$$

$$\int \sin(\ln x)\,dx = \;?$$

integrand je spojitá fce na $(0,+\infty)$

$\rightarrow$ PF bude existovat na $(0,+\infty)$

$$= \left| \begin{array}{c} \text{sub. (2)} \\ x = e^t,\ t \in \mathbb{R} \\ e^t : \mathbb{R} \xrightarrow{\text{na}} (0,+\infty) \\ \frac{d}{dt}e^t \neq 0 \quad \forall t \in \mathbb{R} \\ dx = e^t\,dt \end{array} \right| = \int \sin t\; e^t\,dt =$$

$$= \left| \begin{array}{ll} u = \sin t, & u' = \cos t \\ v' = e^t, & v = e^t \end{array} \right| = e^t \sin t - \int e^t \cos t\,dt$$

$$\int e^t \sin t\,dt = e^t \sin t - \int e^t \cos t\,dt = \left| \begin{matrix} u=\cos t, & u'=-\sin t \\ v'=e^t, & v=e^t \end{matrix} \right|$$

$$= e^t \sin t - e^t \cos t - \int e^t \sin t\,dt$$

$$\Rightarrow$$

$$\int e^t \sin t\,dt = \frac{1}{2} e^t (\sin t - \cos t) + C$$

$$= \frac{x}{2}(\sin \ln x - \cos \ln x) + C$$

$x = e^t$
$t = \ln x$

na $(0, +\infty)$