Limity funkcí

1a) Dokažte z definice

$$\lim_{x\to 1}\left(\frac{x}{2}\right)^3 = \frac{1}{8}$$

$$f(x) = \left(\frac{x}{2}\right)^3 = \frac{x^3}{8}$$

Def $\lim\limits_{x\to 1} f(x) = \frac{1}{8}$ zn. :

x_0

$\forall \varepsilon > 0 \quad \exists \delta > 0 :$

$\forall x \in P(1, \delta) : \quad f(x) \in U\left(\frac{1}{8}, \varepsilon\right)$

Nechť $\varepsilon > 0$.

$\forall \delta > 0 : \quad \forall \Delta x \in \mathbb{R}, \quad |\Delta x| < \delta$

Potom

$$\left| \frac{(1+\Delta x)^3}{8} - \frac{1}{8} \right| =$$

$$= \frac{1}{8} \left| \cancel{1} + 3\Delta x + 3(\Delta x)^2 + (\Delta x)^3 - \cancel{1} \right|$$

$$\leq \frac{1}{8} \left(3\delta + 3\delta^2 + \delta^3 \right)$$

$$\leq \frac{1}{8} \delta \left(3 + 3\delta + \delta^2 \right)$$

Nech $\delta \leq 1$

$$\leq \frac{1}{8} \delta \left(3 + 3 + 1 \right)$$

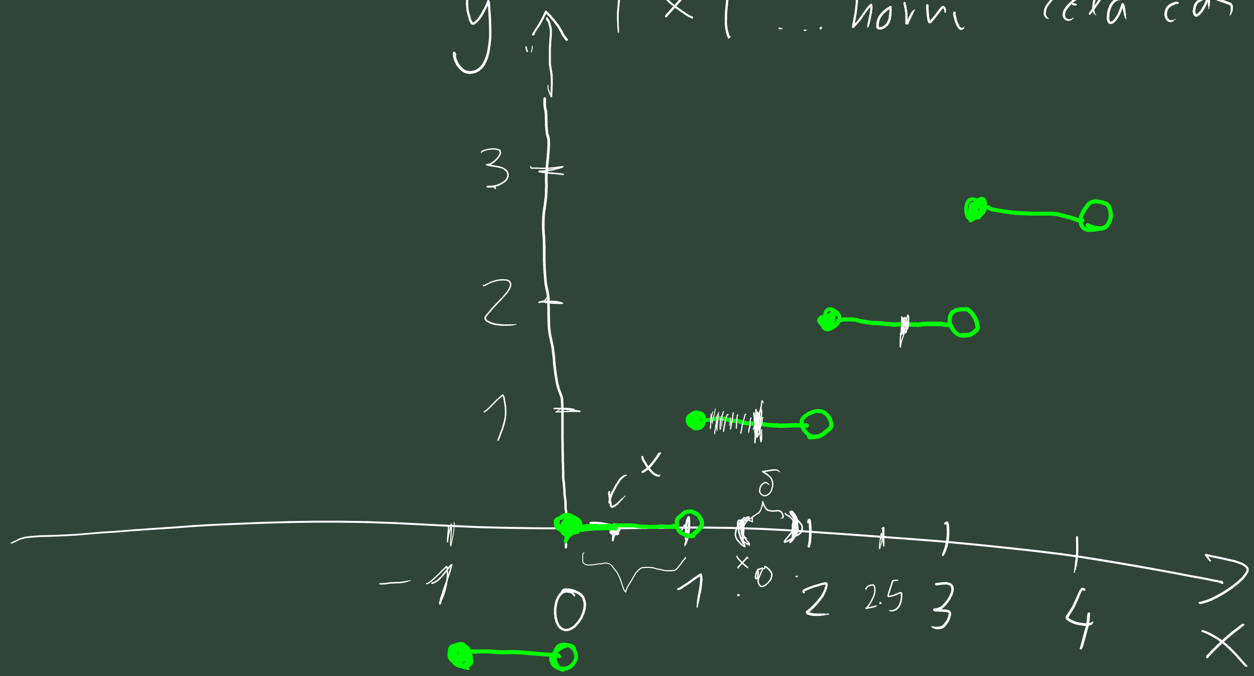
Nech $\delta < 1$

$$= \frac{7}{8} \delta \leq \delta < \varepsilon$$

Stačí zvoliť $\boxed{\delta < \min\{1, \varepsilon\}}$ $\square$

16) $\displaystyle\lim_{x \to 1^+} \lfloor x \rfloor = 1$ — dolní celá část

$\lceil x \rceil$ … horní celá část



Chci:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 :$$

$$\forall x \in P^+(1, \delta) : \quad \left| \lfloor x \rfloor - 1 \right| < \varepsilon$$

$\forall \varepsilon > 0$ Zvolím $\delta = 1$

Protože $P^+(x_0, \delta) = (x_0, x_0 + \delta)$

$\forall x \in P^+(1,1) = (1,2)$

Platí $\lfloor x \rfloor = 1$

Takže

$$|\lfloor x \rfloor - 1| = 0 < \varepsilon$$

$\square$

1c) $\lim\limits_{x \to 1^-} \lfloor x \rfloor = 0$

Analogické 1b)

$x = 1{,}5$

$\lfloor x \rfloor = \lfloor 1{,}5 \rfloor = 1$

1d) spočítejte:

$\lim\limits_{x \to 1} \lfloor x \rfloor = ?$

NEEXISTUJE

2a) $$\lim_{x\to 0} \frac{x^2-1}{\underbrace{2x^2-x-1}_{\text{racionální funkce}}} = ?$$

TRIK: využívám def. spojitosti

$$\lim_{x\to c} f(x) = f(c)$$

spojitá v bodě c

Většina standardních funkcí je spojitá v celém svém def. oboru

$\sqrt{x}$, $|x|$, e^x, $\ln x$, $\cos x$, $\sin x$, $\tan x$, $\arccos x$, $\arcsin x$, $\frac{1}{x}$

f, g : $f+g$, fg, f/g, $f(g(x))$

aritmetika spojitost

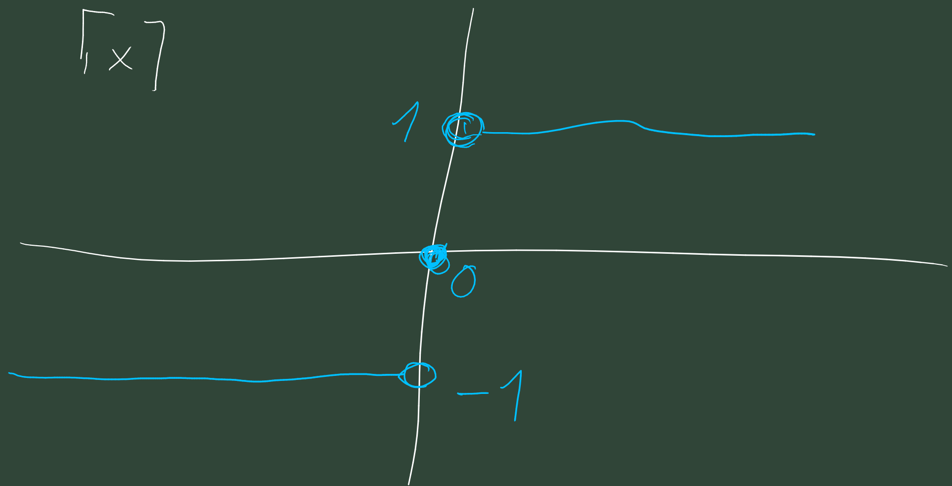
$$\lim_{x \to 0} \frac{x^2 - 1}{2x^2 - x - 1} \overset{\cancel{x}}{=} \frac{0^2 - 1}{2 \cdot 0^2 - 0 - 1} = \frac{-1}{-1} = 1$$

(racionální funkce je spojitá v každém bodě svého def. oboru)

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 1 \end{cases}$$

$\lfloor x \rfloor$

$\lceil x \rceil$



2b)

$$\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} =$$

$$\overset{?}{\neq} \frac{1^2-1}{2-1-1} = \frac{0}{0}$$

nedefinovaný výraz

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(2x+1)} =$$

$1, -\frac{1}{2}$

$$= \lim_{x\to 1} \frac{(x+1)}{2x+1} = \frac{1+1}{2+1} = \frac{2}{3}$$

3) Spočítejte

$$\lim_{x \to 2} \left(\frac{1}{x^2 - 2x} - \frac{x}{x^2 - 4} \right) = ?$$

$$= \lim_{x \to 2} \left(\frac{1}{x(x-2)} - \frac{x}{(x-2)(x+2)} \right)$$

$$= \lim_{x \to 2} \frac{x + 2 - x^2}{x(x-2)(x+2)}$$

$$= \lim_{x \to 2} \frac{-(\cancel{x-2})(x+1)}{x(\cancel{x-2})(x+2)}$$

$$= \lim_{x \to 2} \frac{-(x+1)}{x(x+2)} = -\frac{3}{8}$$

$$(x+y)(x-y) = x^2 - y^2$$

Důležitý trik

$$\sqrt{A} - \sqrt{B} = \left(\sqrt{A} - \sqrt{B}\right) \cdot \frac{\sqrt{A} + \sqrt{B}}{\sqrt{A} + \sqrt{B}} =$$

$$= \frac{A - B}{\sqrt{A} + \sqrt{B}}$$

je definována na $P^+(0, \delta)$

11)
$$\lim_{x \to 0^+} \frac{\sqrt{\frac{1}{x^2} - 1} - \sqrt{\frac{1}{x^2} + 1}}{x} =$$

$$= \lim_{x \to 0^+} \frac{1}{x} \frac{\left(\frac{1}{x^2} - 1\right) - \left(\frac{1}{x^2} + 1\right)}{\sqrt{\frac{1}{x^2} - 1} + \sqrt{\frac{1}{x^2} + 1}} =$$

$$= \lim_{x \to 0^+} \frac{1}{x} \; \frac{-2}{\sqrt{\frac{1}{x^2} - 1} + \sqrt{\frac{1}{x^2} + 1}}$$

$$= \lim_{x \to 0^+} \frac{-2}{\sqrt{x^2\left(\frac{1}{x^2} - 1\right)} + \sqrt{x^2\left(\frac{1}{x^2} + 1\right)}}$$

$$= \lim_{x \to 0} \frac{-2}{\sqrt{1 - x^2} + \sqrt{1 + x^2}} =$$

$$= \frac{-2}{\sqrt{1 - 0^2} + \sqrt{1 + 0^2}} = \frac{-2}{1 + 1} = \underline{\underline{-1}}$$