ON GROUP MODULES

M. TAMER KOŞAN AND JAN ZEMLIČKA

ABSTRACT. The paper is focused on questions when some homological and submodule-chain conditions satisfied by a module $M$ are preserved by the group module $MG$. Namely, it is proved for a group $G$ and an $R$-module $M$ that $MG_{RG}$ is flat if and only if $M_R$ is flat, and $MG_{RG}$ is artinian if and only if $M_R$ is artinian and $G$ is finite, which are two questions raised by Yiqiang Zhou: On Modules Over Group Rings, Noncommutative Rings and Their Applications LENS July 1-4, 2013.

Throughout the paper $R$ will always denote a ring with identity and the notion of an $R$-module will mean a unitary right module. Let us start with the key definition of a group module which generalizes the widely studied notion of a group ring. Suppose that $G$ is a group, and $M$ is a module over a ring $R$. Let $MG$ denote the set all formal linear combinations of the form $\sum_{g \in G} m_g g$, where $m_g \in M$ and $m_g = 0$ for almost all $g$. Denote by $RG$ the corresponding group ring and determine on $MG$ structure of a right $RG$-module:

$$\sum_{g \in G} m_g + \sum_{g \in G} n_g g = \sum_{g \in G} (m_g + n_g) g,$$

$$(\sum_{g \in G} m_g g)(\sum_{g \in G} h_g g) = \sum_{g \in G} \sum_{h, h' : hh' = g} m_h h' g)$$

for all elements $\sum_{g \in G} m_g g, \sum_{g \in G} n_g g \in MG$ and $\sum_{g \in G} r_g g \in RG$. Then the module structure $MG_{RG}$ is correctly defined and it is said to be a group module over the group $G$ by [5]. If we identify every element $m \in M$ with $m \cdot 1 \in MG$, it is easy to see that $M$ is an $R$-submodule of $MG$, where $1$ denotes the identity element of $G$. By [7, Lemma 2.1], if $MG$ is a group module, then $MG \cong_{RG} M \otimes_R RG$.

In [13], Zhou asked the following two questions in his presentation:

Q1. Characterize when $MG_{RG}$ is flat.

Q2. Characterize when $MG_{RG}$ is artinian.

Let $G$ be a group and $M$ be a nonzero $R$-module. In this note, we answer these two questions:

- $M_R$ is a flat $R$-module if and only if $MG_{RG}$ is a flat $RG$-module (see Theorem 8).
- $MG_{RG}$ is artinian if and only if $M_R$ is artinian and $G$ is finite (see Theorem 19).

Furthermore, we prove several necessary conditions of a group under which the group module satisfies some other conditions on chain of submodules, in particular:

- If $MG_{RG}$ is semiartinian, then $M_R$ is semiartinian (see Theorem 11).
- If $MG_{RG}$ is noetherian, then both $M_R$ and $G$ are noetherian (see Theorem 20).

Throughout this article, for a submodule $N$ of $M$, we use $N \leq M$ ($N < M$) to mean that $N$ is a submodule of $M$ (respectively, a proper submodule), and we write $N \leq^e M$ to indicate that $N$ is an essential submodule of $M$. We write $J(R), J(M), \text{Soc}(R), \text{Soc}(M), Z(R)$ for the Jacobson radical of the ring $R$, for the radical of the module $M$, the socle of $R$, the socle of $M$ and the singular ideal of $R$, respectively. For an element $m$ of a module $M$, $r_R(m) = \{ r \in R | mr = 0 \}$ is the annihilator of $m$.

2010 Mathematics Subject Classification. 16D10 (16S50, 16D70).

Key words and phrases. group module, group ring, ads module, excellent extension, artinian module, semiartinian module, noetherian module, flat module.
1. Flat and ADS-modules

Recall that a right module $M$ over a ring $R$ is said to be ADS if for every decomposition $M = A \oplus B$ and every complement $C$ of $A$, we have $M = A \oplus C$ ([11], see also, [6]).

Before we start to investigate group ADS-modules, we need to recall the notion of an excellent extension, introduced by Passman [11], and named by Bonami [2].

Let $R$ and $S$ be rings with the same unity such that $R$ is a subring of $S$. The ring $S$ is an excellent extension of $R$ if the following conditions are satisfied:

1. If $M$ is an $S$-module with an $S$-submodule $S_1$ and $N$ is a direct summand of $M$ as an $R$-module, then $N$ is a direct summand of $M$ as an $S$-module.

2. There is a finite set $\{1 = s_1, s_2, \ldots, s_n\} \subseteq S$ such that $S$ is a free left and right $R$-module with a basis $\{1 = s_1, s_2, \ldots, s_n\}$ and $Rs_i = s_iR$ for all $i = 1, \ldots, n$.

As it is shown in [9], examples of excellent extensions include $n \times n$ matrix rings $M_n(R)$ and crossed product $R * G$, where $G$ is a finite group with $|G|^{-1} R$.

We will need the following facts.

Lemma 1. [1, Lemma 3.1] An $R$-module $M$ is ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are mutually injective.

Lemma 2. [10, Corollary 1.4] Let $S$ be an excellent extension of $R$, and $M$ and $N$ be $S$-modules. If $N_R$ is $M_R$-injective then $N_S$ is $M_S$-injective.

Lemma 3. [10, Lemma 1.5] Let $S$ be an excellent extension of $R$, and $M$ and $N$ be $R$-modules. If $N \otimes_R S$ is $M \otimes_R S$-injective then $N_R$ is $M_R$-injective.

Let us prove several elementary facts on submodules of a group module (cf. [5, 7]). For an $R$-module $M$, $S_R(M)$ denotes the set of all submodules of $M$.

Lemma 4. Let $M$ be a nonzero $R$-module, $G$ a group and $H$ a subgroup of $G$.

1. The functor $- \otimes_{R_H} R_G : RH \to RG$ is exact, preserves direct limits, and $A \otimes_{R_H} RG \neq 0$ for each nonzero $RH$-module $A$.

2. There exists the unique isomorphism $\varphi^H_M : RH \otimes R_H RG \to MG$ satisfying the condition $\varphi^H_M(mh \otimes g) = mhg$ for every $m \in M$, $h \in H$ and $g \in G$.

3. The map $\Phi^H_M : S_{R_H}(MH) \to S_{RG}(MG)$ defined by the rule $\Phi^H_M(A) = \varphi^H_M(A \otimes R_H RG)$, where $A \otimes R_H RG$ is identified with the corresponding submodule of $RH \otimes RH RG$, is injective and monotonic with respect to ordering by inclusion.

Proof. (1) Let $T$ be a right transversal of the subgroup $H$. Then $RG \cong RH \otimes_{\bigoplus_{i \in T}} R_H t$ is a free left $RH$-module. Hence $- \otimes_{R_H} RG$ is exact and $A \otimes_{R_H} RG \cong RH A \otimes_{R_H} RH(t) \cong RH A(t) \neq 0$ for any $A \neq 0$. Moreover, the tensor functor $- \otimes_{R_H} RG$ preserves direct limits by Eilenberg-Watts Theorem because $RG$ is obviously a flat $R$-module.

(2) The existence of the surjective homomorphism $\varphi^H_M$ follows from the universal property of the tensor product. The proof of the injectivity of $\varphi^H_M$ is an easy exercise.

(3) First note that for $A \leq B \leq MH$ we have that $A \otimes R_H RG \leq B \otimes R_H RG \leq MH \otimes R_H RG$. Since we can identify the tensor product $N \otimes R_H RG$ for every $N \leq MH$ with a submodule $\{\sum n_i \otimes \alpha_i | n_i \in N, \alpha_i \in R_H}\$ of the $RG$-module $MH \otimes R_H RG$, and in the sequel which follows by (1)). Since $\varphi^H_M$ is an isomorphism, we obtain

$$\Phi^H_M(A) = \varphi^H_M(A \otimes R_H RG) \leq \varphi^H_M B \otimes R_H RG = \Phi^H_M(B).$$

It remains to prove that $\Phi^H_M$ is injective. Let $A \neq B$. Then either $A \subsetneq A + B$ or $B \subsetneq A + B$. Without loss of the generality, we suppose the strictness of the first inclusion.

Applying the functor $- \otimes_{R_H} RG$ on the exact sequence

$$0 \to A \to A + B \to B + A/A \to 0$$

we get by (1) the exact sequence

$$0 \to A \otimes_{R_H} RG \to (A + B) \otimes_{R_H} RG \to (B + A/A) \otimes_{R_H} RG \to 0.$$
Since the monomorphism \( \alpha \) (identified with the inclusion) is not epimorphism, we have
\[
A \otimes_{RH} (A + B) \otimes_{RH} RG = A \otimes_{RH} RG + B \otimes_{RH} RG,
\]
which proves that \( A \otimes_{RH} RG \neq B \otimes_{RH} RG \). As \( \varphi_M^H \) is an isomorphism,
\[
\Phi_M^H(A) = \varphi_M^H(A \otimes_{RH} RG) \neq \varphi_M^H(B \otimes_{RH} RG) = \Phi_M^H(B)
\]
as desired. \( \Box \)

**Proposition 5.** Let \( S \) be an excellent extension of \( R \) and let \( M \) be a right \( S \)-module.

1. If \( M_R \) is an ADS-module, then so is \( M_S \).
2. If \( M \otimes_R S_S \) is an ADS-module, then so is \( M_R \).

**Proof.** (1) Let \( M = A_S \oplus B_S \). Then \( A_R \) is \( B_R \)-injective by Lemma 1 hence \( A_S \) is \( B_S \)-injective by Lemma 2, which suffices to prove by Lemma 1.

(2) Let \( M = A_R \oplus B_R \). Then \( M \otimes_R S_S = (A_R \otimes_R S_S) \oplus (B_R \otimes_R S_S) \) hence \( A \otimes_R S_S \) is \( A \otimes_R S_S \)-injective by Lemma 1 and so \( A_R \) is \( B_R \)-injective by Lemma 3. Now it remains to use Lemma 1 again. \( \Box \)

**Corollary 6.** Let \( M \) be an \( R \)-module and \( G \) be a finite group with an invertible order in \( R \). If \( MG \) is an ADS \( RG \)-module, then \( MG_R \) is an ADS \( R \)-module.

**Proof.** Since \( MG_R \cong M \otimes_R RG_R \) by Lemma 4(1) and \( RG \) is an excellent extension by [12, Lemma 1.1], we can apply Proposition 5(2). \( \Box \)

As Lemma 3 could be easy generalized for extensions which are excellent relatively to a module \( M \), i.e. such that the second axiom holds only for direct summands of \( M \), we could suppose invertibility of the group order in the ring \( \text{End}(M) \) instead of \( R \).

However the notion of the ADS module naturally generalizes semisimple modules, [5, Theorem 2.3] cannot be directly generalized for an ADS module as:

**Example 7.** (1) Let \( F \) be a field and \( G \) an infinite cyclic group. Then \( FG \cong F[x, x^{-1}] \) is a trivial ADS \( FG \)-module since it is a domain, however \( G \) is infinite.

(2) Let \( p \) be a prime and \( \mathbb{Z}_p \) be a field and \( G \) a group both of order \( p \). Then
\[
\mathbb{Z}_p G \cong \mathbb{Z}_p[x]/(x^p - 1) = \mathbb{Z}_p[x]/(x - 1)^p
\]
is a local ring. Then \( \mathbb{Z}_p G \) is an indecomposable \( \mathbb{Z}_p G \)-module, so it is ADS. However, the order of \( G \) is zero in \( \mathbb{Z}_p \).

Now, we characterize flat group modules.

**Theorem 8.** Let \( G \) be a group and \( M \) an \( R \)-module. Then \( M \) is a flat \( R \)-module if and only if \( MG \) is a flat \( RG \)-module.

**Proof.** (\( \Rightarrow \)) By [8, Theorem 4.34], the module \( M_R \) is a direct limit of a directed system \( (F_i, i \in I) \) consisting of finitely generated free modules. Since \( - \otimes_R RG \) preserves direct limits by Lemma 4(1), \( MG_R \cong M \otimes_R RG_R \) is a direct limit of the directed system \( (F_i \otimes_R RG, i \in I) \) consisting of free \( RG \)-modules, which is flat by [8, Proposition 4.4].

(\( \Leftarrow \)) Applying [8, Theorem 4.34], we get that \( MG_R \) is a direct limit of a directed system \( (M_i, i \in I) \) consisting of finitely generated free \( RG \)-modules. Obviously \( M_i \) are free \( R \)-modules as well and \( (M_i, i \in I) \) is a directed system in the category of \( R \)-modules. Then \( MG_R \) is a direct limit of free modules \( (M_i, i \in I) \) in the category of \( R \)-modules by [4, Lemma 2.3]. Hence \( MG_R \) is flat by [8, Proposition 4.4]. Since \( M_R \) is a direct summand in \( MG_R \), it is flat by [8, Proposition 4.2]. \( \Box \)
2. Modules satisfying some chain conditions

A module $M$ is said to be semiartinian if every non-zero factor of $M$ has a nonzero socle (or, equivalently, each non-zero factor of $M$ contains a simple submodule). Given a semiartinian module $M$, the socle chain of $M$ is a continuous strictly increasing chain $(M_\alpha | \alpha \leq \sigma)$ of submodules of $M$ satisfying $M_{\alpha+1}/M_\alpha = \text{Soc}(M/M_\alpha)$ for each $\alpha < \sigma$ and $M = M_\sigma$. Notice that everyartinian module is semiartinian.

We start the section with an easy technical observation.

**Lemma 9.** Let $M$ be a nonzero $R$-module, $N \le G$, $m \in M \setminus \{0\}$ and $m_1 \in M \setminus N$. If $\text{Soc}(M) = 0$ and $mR \cap N = 0$, then there exists $r \in R$ such that $mr \neq 0$ and $m_1 \notin mrR + N$.

**Proof.** If $m_1 \notin mR + N$, then it suffices to take $r = 1$. Suppose that $m_1 \in mR + N$ and denote by $r$ the canonical projection $M \to M/N$. Let us observe that $\text{Soc}(mR) = 0$ because $\text{Soc}(M) = 0$, and

$$\pi(mR) = \pi(mR) = mR + N/N \cong mR$$

as $mR \cap N = 0$. Since $\overline{0} \neq \pi(m_1) \in \pi(mR)$ and $\text{Soc}(\pi(mR)) = 0$, there exists $\overline{r} \leq \pi(mR)$ such that $\pi(m_1) \notin \overline{r}$. This means that there exists $r \in R$ such that $\overline{0} \neq \pi(m) r \in \overline{r}$. Hence $mr \neq 0$ and $m_1 \notin mrR + N$. \hfill $\Box$

The following claim constitutes a basic step of our prove that semiartinian group modules have semiartinian underlying modules.

**Lemma 10.** Let $M$ be a nonzero $R$-module and $G$ be a group. If $\text{Soc}(MG_{RG}) \neq 0$, then $\text{Soc}(MR) \neq 0$.

**Proof.** If $G = 1$, then $MG \cong M$ and there is nothing to prove. Let $G$ be a nontrivial group and fix an element $m = \sum_{i=1}^n m_i g_i \in \text{Soc}(MG)$ with a minimal $n$ such that $G M R$ is a simple $R G$-module. Note that $m$ is non-zero and $r_{\mu}(m_\alpha) = r_{\mu}(m_\alpha) \neq R$ for all $\alpha < \beta \leq n$, otherwise, if there is $s \in r_{\mu}(m_\alpha) \setminus r_{\mu}(m_\beta)$, then $ms = \sum_{i=1, i \neq a}^n m_i s g_i$ gives an example of a shorter element generating the same simple module.

Assume to contrary that $\text{Soc}(M_\alpha) = 0$. We will show by the induction on $t$ that for every $t = 0, \ldots n$ there exists $s \in R \setminus r_{\mu}(m_1)$ such that $m_1 \notin \sum_{i=1}^t m_s R$. Since $m_1 \neq 0$, the claim is clear for $s = 1$ and $t = 0$. Suppose that there exists $s_{t-1} \in R \setminus r_{\mu}(m_1)$ such that $m_1 \notin \sum_{i=1}^{t-1} m_i s_{t-1} R$. Let us put $N = \sum_{i=1}^{t-1} m_i s_{t-1} R$ and we will prove the claim is true for $t$. If there exists $r \in R$ such that $0 \neq m_i s_{t-1} r \in N$ put $s = s_{t-1} r$ and we are done because $r_{\mu}(m_1) = r_{\mu}(m_t)$. Otherwise suppose $m_i s_{t-1} R \cap N = 0$. As $m_i s_{t-1} \neq 0$ and $m_t s_{t-1} \neq 0$ we may apply Lemma 9, hence there exists $r \in R \setminus r_{\mu}(m_i s_{t-1})$ such that $m_1 \in m_i s_{t-1} r R + N \supset m_i s_{t-1} r R + \sum_{i=1}^t n_i s_{t-1} R$. If we put $s = s_{t-1} r$, then $m_1 \notin \sum_{i=1}^{t-1} m_i s_{t-1} R$. Since $r_{\mu}(m_1) = r_{\mu}(m_t)$, we can see $m_1 s_1 \neq 0$, hence then proof of the induction step is done.

Let $s$ be an element for which $m_1 s_1 \neq 0$ and $m_1 \notin \sum_{i=1}^n m_i s R$. Then $0 \neq ms \in m R G$ so $\text{Soc}(MG) = m R G$ is simple. Hence there exists an element $\rho = \sum_{j} r_j h_j \in RG$ for which $m s R = m R$. Thus $m_1 = \sum_{i,j: g_i = h_j} m_i s r_j$ which contradicts to $m_1 \notin \sum_{i=1}^n m_i s R$. \hfill $\Box$

**Theorem 11.** Let $M$ be an $R$-module and $G$ be a group. If $MG_{RG}$ is semiartinian then $M_R$ is semiartinian.

**Proof.** Let $N$ be an arbitrary proper submodule of $M$. It is enough to show that $\text{Soc}(M/N) \neq 0$. Since $NG$ is a proper submodule of $MG$ and a nonzero factor of a semiartinian module is semiartinian, we get that $MG/NG \cong (M/N)G$ has an essential socle. Hence $\text{Soc}(M/N)$ is nonzero by Lemma 10. \hfill $\Box$

Note that Example 7(1) shows that for an infinite cyclic group $G$ and a field $F$, the $FG$-module $FG$ is not semiartinian however $F$ is even artinian.

Using a result of the work [5] about semisimple group modules, we characterize semiartinian group modules over finite groups having invertible order in its endomorphism ring.
Proposition 12. Let $M$ be an $R$-module and $G$ be a finite group with order invertible in $\text{End}_R(M)$. Then $M_R$ is semiartinian if and only if $MG_{RG}$ is semiartinian.

Proof. Suppose that $M_R$ is semiartinian with the socle chain $(M_\alpha | \alpha \leq \sigma)$. Since $M_{\alpha+1}/M_\alpha$ is a semisimple $R$-module, $M_{\alpha+1}G_{RG}/M_\alpha G_{RG} \cong (M_{\alpha+1}/M_\alpha)G_{RG}$ is a semisimple $RG$-module by [5, Theorem 3.2] for every $\alpha < \sigma$. Thus $MG_{RG}$ is semiartinian.

If, on the other hand, $MG_{RG}$ is a semiartinian $RG$-module, then $M_R$ is a semiartinian $R$-module by Theorem 11.

The following claim shows that several constructions of non-artinian group rings work also in the case of group modules.

Proposition 13. Let $M$ be a nonzero $R$-module and $G$ be a group. If

1. either $G$ is an infinite cyclic group
2. or $G$ contains an infinite strictly increasing chain of finite subgroups,

then $MG_{RG}$ is not artinian.

Proof. (1) Let $g$ be a generator of a cyclic group $G$ and $m \in M \setminus \{0\}$. Define a cyclic submodule $M_n = m(1 + g)^nRG$ for every $n$. Then $M_1 \supseteq M_2 \supseteq \ldots$ forms a decreasing chain of submodules and it remains to prove that $M_n \supsetneq M_{n+1}$ for every $n$.

Assume that there exists $n$ such that $M_n = M_{n+1}$. There are integers $u,v$ and $\alpha = \sum_{i=u}^v a_ig^i \in RG$ such that $u \leq v$, $ma_v \neq 0 \neq ma_u$ and

$$m(1 + g)^n = m(1 + g)^{n+1} = m(1 + g)^n(a_ug^u + \sum_{i=u+1}^v (a_i + a_{i-1})g^i + a_vg^{v+1}).$$

Comparing coefficients of $g^u$ in case that $u < 0$ we obtain that $ma_u = 0$, a contradiction. If $u \geq 0$, then $v \geq u \geq 0$, and comparing coefficients of $g^{u+v+1}$ we get equality $ma_v = 0$, which contradicts to chose of $\alpha$.

Since $M_1 \supseteq M_2 \supseteq \ldots$ is a strictly decreasing chain of submodules, $MG$ is not an artinian $RG$-module.

(2) Let $H_1 \subseteq H_2 \subseteq \ldots$ be a strictly increasing chain finite subgroups of $G$ and $m \in M \setminus \{0\}$. Put $\gamma_i = \sum_{h \in H_i} mh$ for each $i$. If $T$ is a right transversal of the subgroup $H_i$ in the group $H_{i+1}$, then $\gamma_{i+1} = \gamma_i \sum_{t \in T} 1t$, which proves that $\gamma_{i+1} \in \gamma_iRG$. Furthermore, if $\sum g_hg \in \gamma_{i+1}RG$, then $m_1 = mh$ for every $h \in H_{i+1}$. Since $H_i \subseteq H_{i+1}$ we see that $\gamma_i \notin \gamma_{i+1}RG$. We have constructed a strictly decreasing chain of submodules $\gamma_1RG \supsetneq \gamma_2RG \supsetneq \ldots$ which witnesses that $MG$ is not artinian.

Recall that a group $G$ is called locally finite if every finitely generated subgroup of $G$ is finite and $G$ is periodic if all its elements have a finite order.

Example 14. (1) Let $G = \mathbb{Z}_p^\infty$ be a Prüfer $p$-group for a prime $p$. Then $G$ is a periodic artinian group and $MG$ is non-artinian for every nonzero artinian module $M$ by Proposition 13(2).

(2) If $G$ is an infinite locally finite group, it contains an infinite set $\{g_i|i \in \mathbb{N}\} \subseteq G$ such that $g_n \notin \langle g_1, \ldots, g_{n-1} \rangle$ for each $n$. Then $H_i = \langle g_1, \ldots, g_i \rangle$, $i \in \mathbb{N}$ forms an infinite strictly increasing chain of finite subgroups, so $MG_{RG}$ is non-artinian by Proposition 13(2) for an arbitrary nonzero module $M$. In particular, if $G = \mathbb{Q}/\mathbb{Z}$, we can see that the structure of decreasing chains of submodules is very reach by Lemma 4.

(3) If $G$ contains an infinite cyclic subgroup $\langle g \rangle$, then $M\langle g \rangle_{R\langle g \rangle}$ is non-artinian by Proposition 13(1), hence we can find an strictly decreasing chain of submodules in $MG_{RG}$ by Lemma 4(3).

The following observation is a straightforward consequence of Lemma 4.

Lemma 15. Let $M$ be an $R$-module, $G$ a group and $H$ a subgroup of $G$. If $MG$ is artinian (noetherian), then $MH$ is artinian (noetherian) as well.

Proof. If $M = 0$, there is nothing to prove. If $\langle A_i | i \in \mathbb{N} \rangle$ is a strictly decreasing (increasing) chain of submodules of $MH$, then we have $\langle \Phi^i_{\langle A \rangle} | i \in \mathbb{N} \rangle$ forms a strictly decreasing (increasing) chain of submodules of $MG$ by Lemma 4(3).
Proposition 16. Let $M$ be an $R$-module and $G$ be a group.

(1) If $M$ is artinian (noetherian) and $G$ is finite, then $MG_{RG}$ is artinian (noetherian).

(2) If $MG_{RG}$ is artinian then $M_R$ is artinian and $G$ is periodic.

Proof. (1) Since $MG \cong_R M^{[G]}$ is an artinian (noetherian) $R$-module, it is also artinian (noetherian) as an $RG$-module.

(2) Note that $M(g)$ is an artinian $R(g)$-module for each $g \in G$ by Lemma 15, in particular $M \cong_R M(1)$ is an artinian $R$-module. Since $M(g)$ is artinian, the cyclic group $(g)$ is finite by Proposition 13(1), which proves that $G$ is periodic. □

It is well known that if $e \in R$ is an idempotent and $M$ is an $R$-module, then $e$ is identity of the unitary ring $eRe$ and $Me$ has a natural structure of $eRe$-module.

Lemma 17. Let $e \in R$ be an idempotent, $M$ a nonzero $R$-module and $G$ a group.

(1) If $Ke$ and $Le$ are $eRe$-submodules of the module $Me$ such that $K \subseteq L$, then $KeR$ and $LeR$ are $R$-submodules of $M$ and $KeR \subseteq LeR$.

(2) If $M$ is an artinian (noetherian) $R$-module, then $Me$ is an artinian (noetherian) $eRe$-module.

(3) If $MG$ is an artinian (noetherian) $RG$-module, then $MeG$ is an artinian (noetherian) $eReG$-module.

Proof. (1) As $K \subseteq L$, we obtain that $KeR \subseteq LeR$ are submodules of the $R$-module $M$. Assume that $KeR = LeR$. Then $K = KeRe = LeRe = L$, which contradicts to the hypothesis $K \subseteq L$.

(2) If $(N_i)_{i \in \mathbb{N}}$ is a strictly decreasing (increasing) chain of $eRe$-submodules of the module $Me$, then $(N_iR)_{i \in \mathbb{N}}$ forms a strictly decreasing (increasing) chain of $R$-submodules of $M$ by (1).

(3) Since $R$ is a subring of the group ring $RG$, the element $e$ is an idempotent of $RG$. Furthermore $e$ commutes with all elements of $G$, hence $MG = MeG$ is a module over $eRG = eReG$. Now the claim follows from (2). □

The key role in our main result presents the following translation of an artinian or noetherian group module over simple module to a construction of an artinian or noetherian group ring.

Proposition 18. Let $S$ be a simple $R$-module, $G$ be a group and $T = \text{End}(S_R)$. Then $T$ is a skew-field and

(1) if $SG$ is an artinian $RG$-module, then $TG$ is a right artinian ring,

(2) if $SG$ is a noetherian $RG$-module, then $TG$ is a right noetherian ring.

Proof. Since $S$ is simple, it is easy to see that $T$ is a skew-field, hence $T S \cong_T T^{(\kappa)}$ for some cardinal number $\kappa$ has the structure of a free left $T$-module, i.e. of a vector space over the skew-field $T$. Put $A = \text{End}(T S) \cong \text{End}(T^{(\kappa)})$. Then there exists an idempotent $e \in A$ such that $eAe \cong T$ (any endomorphism which performs as identity on some one-dimensional subspace and it is zero on some complements). Note that $S$ has the structure of the $A$-module and $R$ can be seen as a subring of $A$, so $RG$ is also a subring of $AG$. Since $S$ is a simple $R$-module, it is a simple $A$-module. Moreover, as $SG$ is an artinian (noetherian) $RG$-module, it is an artinian (noetherian) $AG$-module. Now, by Lemma 17(3), $SeG$ is an artinian (noetherian) $eAeG$-module. As $eAe \cong T$ and as $Se$ is a simple module over $eAe$, we obtain that $TG$ is an artinian (noetherian) $TG$ module, which finishes the proof. □

The previous proposition allows us to translate celebrated Connels’ results on chain conditions of group rings [3] to the case of group modules.

Theorem 19. Let $R$ be a ring, $G$ a group, and $M$ be a nonzero $R$-module. Then $MG_{GR}$ is artinian if and only if $M_R$ is artinian and $G$ is finite.

Proof. Note that the reverse implication follows immediately from Proposition 16(1). Suppose that $MG_{GR}$ is artinian. Then $M_R$ is artinian by Proposition 16(2), so it remains to prove that $G$ is finite. Let $S \subseteq M$ be a simple submodule of $M$. Then $SG$ is a submodule of $MG$, hence artinian module. Then $T = \text{End}(S_R)$ is a skew-field for which $TG$ is a right artinian ring by Proposition 18. Hence $G$ is finite by [3, Theorem 1]. □
We say that a group $G$ is noetherian if it satisfies ACC on subgroups.

**Theorem 20.** Let $R$ be a ring, $G$ a group, and $M$ a nonzero $R$-module. If $MG_R$ is noetherian, then both $M_R$ and $G$ are noetherian.

**Proof.** The module $M_R$ is noetherian by Proposition 16(1). Thus there exists a maximal submodule $N \leq M$ and $S = M/N$ is a simple $R$-module. As $MG_R$ is noetherian, the module $SG \cong MG/NG$ is noetherian as well. Applying Proposition 18 again we get that $TG$ is a right noetherian group ring for a skew-field $T = \text{End}(S_R)$. Now, the claim follows from [3, Theorem 2(b)].

We finish the paper by listing several corresponding open problems from which the formulation of the third one is due to Zhou [13] and the last one is for long time open even in context of group rings:

**Question.** Describe equivalent conditions on a module $M$ and a group $G$ under which $MG$ is semiartinian, ADS, pure injective, or noetherian.

**REFERENCES**


**Department of Mathematics, Gazi University, Ankara, Turkey**

*E-mail address: tkosan@gmail.com*

**Department of Algebra, Charles University in Prague, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8, Czech Republic**

*E-mail address: zemlicka@karlin.mff.cuni.cz*