ON FUSIBLE RINGS

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Abstract. A ring R is called left fusible if every nonzero element is the sum of a left zero-divisor and a non-left zero-divisor, and R is called uniquely left fusible if for any $a \in R$ there exists a unique left zero-divisor z such that $a-z$ is non-left zero-divisor. We show that a left fusible ring R is uniquely left fusible if and only if either R is a domain or R has a unique non-left zero-divisor element.

1. Introduction

Throughout this note, R denotes an associative ring with identity. $\mathrm{lann}_R(S)$ denotes the left annihilator of S in R for any nonempty subset $S \subseteq R$. An element $a \in R$ is a *left (right) zero-divisor* if $\mathrm{lann}_R(a) = \{x \in R \mid xa = 0\} \neq 0$ ($\mathrm{rann}_R(a) \neq 0$), $a \in R$ is called *left (right) regular* if it is not a left (right) zero-divisor and a is *regular* if it left and right regular. Let $Z_l(R)$ (respectively, $Z_l^*(R)$) denote the set of left zero-divisors (respectively, left regular elements) of R.

Recall that the sum of two-zero divisors need not be a zero-divisor. Faith and Pillay characterized in [3, Theorem 1.12] those commutative rings for which the set of zero divisors is an ideal. Now, as it is well-known, if the set of left zero-divisors in a ring R is not a left ideal, then there exists a left zero-divisor which can be expressed as the sum of a left zero-divisor and a left regular element in R. This fact motivated Ghashghaei and McGovern [4] to investigate the class of rings in which every element can be written as the sum of a left zero-divisor and a left regular element: a non-zero element $a \in R$ is *left fusible*, if it can be expressed as the sum of a left zero-divisor and a left regular element in R. R is said to be *left fusible*, if every non-zero element of R is left fusible. Right fusible rings are defined analogously. A ring R which is both right and left fusible is called a *fusible ring* (see also [7]). Every left regular element has a trivial left fusible representation given by $r = 0 + r$. Hence, every domain is a fusible ring.

Observe that a left domain has the property that each nonzero element has a unique left fusible representation. A ring for which every nonzero element has a

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unique left fusible representation is called *uniquely left fusible*. In [4, Theorem 2.25], the authors provided a partial characterization of uniquely fusible rings. They have shown that if R is a ring with $0 \neq 2$, then R is a uniquely left fusible ring if and only if R is a uniquely right fusible ring, which is true if and only if R is a domain. In this paper, we investigate this property and complete the characterization of such rings.

2. Uniquely fusible rings

Before we formulate the result from the paper [4] let us recall that a ring R is called *uniquely left fusible* if for any nonzero $a \in R$ there exists a unique left zero-divisor $z \in Z_l(R)$ such that $a - z \in Z_l^*(R)$.

Theorem 2.1. [4, Theorem 2.25] *The following statements are equivalent for a ring R with $2 \neq 0$.*

(1) *R is a uniquely left fusible ring.*
(2) *R is a domain.*
(3) $Z_l(R) = 0$
(4) $Z_r(R) = 0$
(5) *R is a uniquely right fusible ring.*

Now, we consider the remaining case of rings of characteristics 2.

Theorem 2.2. *The following statements are equivalent for a ring R with $2 = 0$.*

(1) *R is a uniquely left fusible ring.*
(2) *Either R is a domain or $Z_l^*(R) = \{1\}$.*
(3) *Either R is a domain or $r(R) = U(R) = \{1\}$.*
(4) *Either R is a domain or $Z_r^*(R) = \{1\}$.*
(5) *R is a uniquely right fusible ring.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is neither a domain nor $Z_l^*(R) = \{1\}$. Hence there exists $0 \neq a \in Z_l(R)$ and $1 \neq b \in Z_l^*(R)$. Note that $a + b \neq 0$.

Since any left regular element $c \in Z_l^*(R)$ has a trivial fusible decomposition $c = 0 + c$ and $a + b$ has a non-trivial fusible decomposition, we get that $a + b \in Z_l(R)$. Similarly $(a + b) + 1$ is a non-trivial fusible decomposition of the nonzero element $a + b + 1$, hence $a + b + 1 \in Z_l(R)$ as well. Thus $a + b = (a + b + 1) + 1$ has two distinct fusible decompositions, a contradiction with the hypothesis that R is uniquely left fusible.

(2) $\Rightarrow$ (1) It is clear.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Henriksen [6, Theorem 2.4] proved that a ring R has a unique left regular element if and only if R has a unique right regular element if and only if R has unique regular element.
(4) $\Leftrightarrow$ (5) The proof is a right-hand version of (1) $\Leftrightarrow$ (2). □

Recall that a commutative ring R is called a *0-ring* if every element different from 1 is a zero-divisor [2]. We note that 0-rings are also known as Cohn's rings, see for example [8]. It is clear that every Boolean ring with a unit-element is a 0-ring, and [2, Theorem 1] shows that for every ring R there exists its extension S such that every element of S which is not invertible in R is a zero-divisor. This extension of the polynomial ring $\mathbb{F}_2[x]$ over the two-element Galois field presents an example a 0-ring which is not Boolean, which answers a question raised by Kaplansky. This fact implies that a commutative ring R with the property $R = Idem(R) \bigcup Z(R)$ may not be Boolean. Hence we conclude that the Anderson and Badawi's conjecture is false, see [1, Page 1022]. In fact, $R = Idem(R) \bigcup Z(R)$ if and only if R is 0-ring.

Let us say more, Henriksen [6] introduced the concept of *UR-rings*, rings with a unique regular element without assuming commutativity, and generalized the concept of 0-rings. Observe that UR-rings are left fusible: for each $a \neq 0, 1$, $a = 1 + (a - 1)$ is a left fusible representation of a.

Now we can formulate an immediate consequence of Theorems 2.1 and 2.2:

Corollary 2.3. *The following statements are equivalent for a ring R.*

1. *R is uniquely fusible.*
2. *R is either a domain or a UR-ring.*

Finally note that a commutative ring R is uniquely fusible if and only if R is either a domain or a 0-ring.

We say that an element has a $(*)$-*representation* if it can be expressed as the sum of a left zero-divisor and two left regular elements.

Clearly, a left zero-divisor is trivially has the $(*)$-representation: For $z \in Z_l(R)$, we get $z = z + 1 + (-1)$. Also if $2 \in U(R)$, then every left regular element n of a ring R has the $(*)$-representation $n = 0 + \frac{n}{2} + \frac{n}{2}$. Thus any element of a ring R with $2 \in U(R)$ has the $(*)$-representation. On the other hand, $\mathbb{Z}_2$ is a fusible ring but not every element has the $(*)$-representation. Let us formulate an easy characterization of rings R such that every element of R has a $(*)$-representation.

Proposition 2.4. *The following statements are equivalent for a ring R.*

(1) *Every element of R has a $(*)$-representation,*
(2) *1 has a $(*)$-representation,*
(3) *there exists a left regular element with a $(*)$-representation,*
(4) *there exists left regular elements x, y, z for which $x + y + z$ is a left zero-divisor.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) The implications are clear
(3) $\Rightarrow$ (4) This follows from the observation that a $(*)$-representation of any left regular element $x = a + b + d$ where $a, b \in Z_l^*(R)$ and $d \in Z_l(R)$, yields a left zero-divisor $x + (-a) + (-b) = d$ as a sum of three left regular element.
(4) $\Rightarrow$ (1) Let $x, y, z \in Z_l^*(R)$ such that $d = x + y + z \in Z_l(R)$ and suppose that $a \in R$. If $a-z+d \in Z_l^*(R)$, then $a = (a-z+d)+z-d$ presents a $(*)$-representation of a. On the other hand, if $a - z + d \in Z_l(R)$, then $a = (a - z + d) - x - y$ is a $(*)$-representation of a. □

We recall that a ring R is called *2-good* if every element is the sum of two units (2-good rings are also known as *2-sum rings*). Clearly, any element of a 2-good ring having the $(*)$-representation. Let F be any field, and $A = F[[x]]$ the power series ring in one variable over A. Let K be the field of fractions of A. The ideal (x^n) denotes all the ideals of A generated by a power of x. The ring

$$R = \{f \in End_F(A) \mid \exists q \in K, n \in \mathbb{N} : \ \mathrm{lann}_R(a) = aq \ \forall \ a \in (x^n)\}$$

has the $(*)$-representation but it is not a 2-good ring (for details see Bergman's example [5, Example 1]).

Theorem 2.5. *The following statements are equivalent for a left fusible ring R.*

(1) *Every element of R has a $(*)$ representation.*
(2) *R has not a unique left regular element (i.e., $Z_l^*(R) \neq \{1\}$).*
(3) *R is not a UR-ring.*

Proof. (1) $\Rightarrow$ (2) By Proposition 2.4, there exist $x, y, z \in Z_l^*(R)$ such that $d = x+y+z \in Z_l(R)$. Hence $x+y = 0$ implies that $z = -d$, which is a contradiction. Now we get two distinct left regular elements x and $-y$, as desired.
(2) $\Rightarrow$ (1) Let $x \neq 1$ be a left regular element. Then the nonzero element $x - 1$ has a fusible presentation $1 - x = a + d$ for $a \in Z_l^*(R)$ and $d \in Z_l^*(R)$. Thus $1 = x + a + d$ is a $(*)$-representation of 1 and so every element of R has a $(*)$-representation by Proposition 2.4.
(2) $\Leftrightarrow$ (3) This is straightforward with the definition. □

Corollary 2.6. *Every element of a commutative fusible ring R having the $(*)$-representation if and only if R is not a 0-ring.*

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