ESSENTIALLY ADS MODULES AND RINGS

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ABSTRACT. This paper introduces the notion of essentially ADS (e-ADS) modules. Basic structural properties and examples of e-ADS modules are presented. In particular, it is proved that (1) The class of all e-ADS modules properly contains all ADS as well as automorphism invariant modules. e-ADS modules serves also as a tool for characterization of various classes of rings. It is shown that: (2) R is a QF-ring if and only if every projective right R-module is e-ADS; (3) R is a semisimple Artinian ring if and only if every e-ADS module is injective. The final part of this paper describes properties of e-ADS rings, which allow to prove a criterion of e-ADS modules for non-singular rings: (4) Let R be a right non-singular ring and Q be its the right maximal ring of quotients. Then R is a right e-ADS ring if and only if either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong \mathbb{M}_2(A)$ for a suitable right automorphism invariant ring A.

1. Introduction

The absolute direct summand (ADS) property for modules was introduced by Fuchs in [6] and recently was intensively studied by Alahmadi, Jain and Leroy in [1]. Recall that a right module M over a ring R is said to be ADS if for every decomposition $M = S \oplus T$ and every complement T' of S, we have $M = S \oplus T'$.

In recent works [5, 8, 12], the notion of automorphism invariant modules was shown to be and important tool for finding correspondences between various concept of injectivity. A module M is called automorphism invariant if it is invariant under automorphisms of its injective hull, equivalently if every isomorphism between two essential submodules of M extends to an automorphism of M [8]. Quasi-injective modules are automorphism invariant. Assume that an R-module M has a decomposition $M = S \oplus T$ such that T' is a complement of S, $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$. It is easy to see (cf. Lemma 2.4) that $E(S) \cong E(T)$, where E denotes the injective hull. In light of this observation, we define essentially ADS-modules (shortly e-ADS), as an R-module M such that for every decomposition $M = S \oplus T$ of M and every complement T' of S with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$, we have $M = S \oplus T'$. This definition naturally generalizes both notions mentioned above. Furthermore, recall that when a module M is quasi-continuous, for each decomposition $M = A \oplus B$, A and B are relatively injective. This property of modules is known to be equivalent to ADS modules ([1, Lemma 3.1]) and automorphism invariant modules ([8, Theorem 5]). Since an R-module M is e-ADS if and only if for each decomposition $M = A \oplus B$, A and B are relatively automorphism invariant (see Lemma 2.8), e-ADS modules arise as a

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generalization of quasi-continuous modules, ADS modules as well as automorphism invariant modules. The goal of this paper is to present a list of significant structural properties of e-ADS modules and to exhibit relations with other notions. We show that e-ADS as well as automorphism invariant or ADS modules have a description in the language of the lattice theory (Lemmas 2.1, 2.15, 4.4). Important for further study is the division of the class of e-ADS modules into trivial and non-trivial case (cf. Theorem 2.9). Moreover it is proved in Theorem 2.9 that, if $E(A) \ncong E(B)$ for each decomposition $M = A \oplus B$, then M is e-ADS. On the other hand if M is an e-ADS module with a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then $A \cong B$ and the modules A and B are automorphism invariant. This result is key to our work and is used to characterize many well-known classes of modules in terms of e-ADS modules. For example, we show in Theorem 2.18 that for an e-ADS module M with a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, M satisfies the exchange property if and only if $End(M)$ is semiregular.

The final part of the article is devoted to rings which are e-ADS as right modules over themselves. By applying elementary lattice theoretical tools on rings induced by idempotents we characterize when non-singular rings are e-ADS. Based on the key observation that a non-trivial e-ADS ring is isomorphic to a 2×2 full matrix ring over an automorphism invariant ring (Lemma 4.9) we prove a characterization of non-singular e-ADS rings. Namely, a non-singular ring is e-ADS if and only if it is either trivial e-ADS or it is a product of a self-injective ring and a matrix ring $M_2(S)$ over an automorphism invariant ring S with many central idempotents (Theorem 4.11).

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. For a submodule N of M, we use $N \leq M$ ($N < M$) to mean that N is a submodule of M (respectively, proper submodule), and we write $N \leq^e M$ to indicate that N is an essential submodule of M. For any term not defined here the reader is referred to [2], [4] and [9].

2. e-ADS modules

Let M and N be two modules. The module M is called automorphism N-invariant if for any essential submodule A of N, any essential monomorphism from A to M can be extended to some $g \in Hom(N, M)$ ([12]).

We note that M is automorphism invariant if M is automorphism M-invariant by [8, Theorem 2].

Lemma 2.1. *Let M and N be modules and $X = M \oplus N$. The following conditions are equivalent:*

1. *M is automorphism N-invariant.*
2. *For any complement K of M in X with $K \cap N = 0$ and $M \cap (K \oplus N) \leq^e M$, the module X has a decomposition $X = M \oplus K$.*

Proof. Consider the natural projections $\pi_M : X \to M$ and $\pi_N : X \to N$. Note that $\pi_M(K) = M \cap (K + N)$ for each submodule K of X.

(1) $\Rightarrow$ (2) Let K be a complement of M in X with $K \cap N = 0$ and $\pi_M(K) \leq^e M$. Clearly, $M \oplus K = M \oplus \pi_N(K)$ so that $\pi_N(K)$ is essential in N. Consider the homomorphism $\theta : \pi_N(K) \to \pi_M(K)$ defined by $\theta(n) = m$ whenever $k = m+n \in K$ for $k \in K, m \in M, n \in N$. It is easy to see that θ is an isomorphism ($K \cap N = 0$ by the assumption). Since M is automorphism N-invariant, the homomorphism θ can

be extended to some $g : N \to M$. Set $T := \{n + g(n) | n \in N\}$. Clearly, $M \oplus T = X$ and T contains K essentially by modularity. Since K is a complement, we obtain $T = K$.

(2) $\Rightarrow$ (1) Let A be an essential submodule of N and $f : A \to M$ be an essential monomorphism. Set $H := \{a - f(a)| : a \in A\}$. Clearly, $H \cap N = 0$, $H \cap M = 0$ and $\pi_M(H) = f(A)$ is essential in M. Then $M \oplus H = M \oplus \pi_N(H) = M \oplus A$, which is essential in X. Let K be a complement of M in X containing H. Then $H \leq^e K$. Hence $K \cap N = 0$ because $H \cap N = 0$. Moreover, $\pi_M(H) \leq \pi_M(K)$ which implies that $\pi_M(K) \leq^e M$. By the assumption, we have $M \oplus K = X$. Now let $\pi : M \oplus K \to M$ be the projection. Then writing an element $a \in A$ in the form $a = a - f(a) + f(a)$, the restriction of π to N is the desired extension of f. □

Lemma 2.2 ([12, Theorem 2.2]). *The following are equivalent for modules M and N:*

(1) *M is automorphism N-invariant.*
(2) *$\alpha(N) \leq M$ for every isomorphism $\alpha : E(N) \to E(M)$.*

As an immediate consequence of Lemmas 2.1 and 2.2 , we obtain the following observation.

Corollary 2.3. If M and N are relatively automorphism invariant modules and $E(M) \cong E(N)$, then $M \cong N$

Lemma 2.4. *Let M be a module with a decomposition $M = S \oplus T$. If T' is a complement of S with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$, then $T \oplus T' \leq^e M$ and $E(S) \cong E(T)$.*

Proof. Note that $S \oplus T' \leq^e M$ because T' is a complement of S. Since

$$T \oplus [S \cap (T' \oplus T)] \subseteq T \oplus T' \text{ and } T \oplus [S \cap (T' \oplus T)] \leq^e T \oplus S = M,$$

we get $T \oplus T' \leq^e M$. Moreover, the injective hulls $E(S)$, $E(T)$ and $E(T')$ can be taken as submodules of the injective hull $E(M)$ such that $S \leq^e E(S)$, $T \leq^e E(T)$ and $T' \leq^e E(T')$. Since $S \cap T' = 0 = T \cap T'$, it is easy to see that

$$E(S) \cap E(T') = 0 = E(T) \cap E(T').$$

On the other hand,

$$E(S) + E(T') = E(M) = E(T) + E(T')$$

because both $E(S) + E(T') = E(S) \oplus E(T')$ and $E(T) + E(T') = E(T) \oplus E(T')$ are injective submodules of $E(M)$, and both $S \oplus T'$ and $T \oplus T'$ are essential in $E(M)$. Thus

$$\begin{aligned} E(T) \cong (E(T) + E(T'))/E(T') &= E(M)/E(T') \\ &= (E(S) + E(T'))/E(T') \cong E(S). \end{aligned}$$

□

In light of Lemma 2.4, we call M an essentially ADS-module, shortly e-ADS, if for every decomposition $M = S \oplus T$ of M and every complement T' of S with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$, we have $M = S \oplus T'$.

Clearly, ADS-modules are e-ADS. The following examples show that the converse is not true in general and that the class of e-ADS modules is not closed under taking direct summands, respectively.

Example 2.5. Let T be a torsion abelian group which is not divisible and put $M := \mathbb{Z} \oplus T$. Then every decomposition $M = A \oplus B$ contains a subgroup which is isomorphic to $\mathbb{Z}$ while the second is torsion, hence $E(A) \ncong E(B)$. By Lemma 2.4, there exists no a decomposition satisfying the hypothesis of the definition of e-ADS exists. So that the conditions for having an e-ADS modules are vacuously satisfied and the module is e-ADS. Hence it is an e-ADS abelian group.

On the other hand, since T is not divisible, we obtain T is not $\mathbb{Z}$-injective and so M is not ADS by [1, Lemma 3.1].

Example 2.6. Put $M := \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ for some prime number p. Then M is e-ADS by Example 2.5. Since $\mathbb{Z}_p$ is not automorphism $\mathbb{Z}_{p^2}$-invariant, we obtain that $Z_p \oplus \mathbb{Z}_{p^2}$ is not e-ADS by Lemma 2.8.

Let us mention the following equivalent conditions for a module to be e-ADS.

Theorem 2.7. *The following conditions are equivalent for a module M:*

(1) *M is e-ADS.*
(2) *For every decomposition $M = S \oplus T$, if T' is a complement of S in M and T is a complement of T' in M, then $M = S \oplus T'$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $M = S \oplus T$ is a decomposition of M, T' is complement of S and T is a complement of T' in M. Then $S \cap (T' \oplus T) \leq^e S$ since $T \oplus T' \leq^e M$. By (1), we have $M = S \oplus T'$.

(2) $\Rightarrow$ (1) Let $M = S \oplus T$ of M and $S \cap (T' \oplus T) \leq^e S$ for a complement T' of S with $T' \cap T = 0$. By Lemma 2.4, $T \oplus T' \leq^e M$. Since T is a direct summand of M, we get T is a complement of T' in M. By (2), we have $M = S \oplus T'$. $\square$

In [1, Lemma 3.1], it is shown that an R-module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.

Lemma 2.8. *An R-module M is e-ADS if and only if for each decomposition $M = A \oplus B$, A and B are relatively automorphism invariant.*

Proof. This is clear from Lemma 2.1. $\square$

The following characterization proves to be quite useful.

Theorem 2.9. *Let M be an R-module.*

(1) *If $E(A) \ncong E(B)$ for each decomposition $M = A \oplus B$, then M is e-ADS.*
(2) *If M is an e-ADS module with a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then $A \cong B$ and the modules A and B are automorphism invariant.*

Proof. (1) This follows from Lemmas 2.2 and 2.8.

(2) By Lemma 2.8 and Corollary 2.3, we have $A \cong B$. Thus A is automorphism A-invariant, i.e. automorphism invariant. $\square$

In the following observation, we continue to obtain equivalent conditions for a module to be e-ADS.

Theorem 2.10. *The following conditions are equivalent for a module M:*

1. *M is e-ADS,*
2. *Assume that M has a decomposition $M = A \oplus B$. For any isomorphism $f \in Hom(E(B), E(A))$, the module M has a decomposition $M = A \oplus X$, where $X = \{b + f(b) |\ b \in B, f(b) \in A\}$.*
3. *For every decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, the module $A \cong B$ is automorphism invariant.*
4. *Either $E(A) \ncong E(B)$ for every decomposition $M = A \oplus B$ or there exists an automorphism invariant module X for which $M \cong X \oplus X$ and for every two decompositions $X = P_1 \oplus Q_1 = P_2 \oplus Q_2$ with $E(P_1) \oplus E(P_2) \cong E(Q_1) \oplus E(Q_2)$ we have $(P_1 \oplus P_2) \cong (Q_1 \oplus Q_2)$ is automorphism invariant.*

Proof. (1) $\Rightarrow$ (2) We show that $X = \{b + f(b)|b \in B, f(b) \in A\}$ is a complement of A in M. Notice that $A \cap X = 0$, $X \cap B = 0$ and $A \cap (X \oplus B) \leq^e A$. Let L be a submodule of M such that $L \cap A = 0$ and $X \leq L$. Consider the natural projections π_A and π_B of M onto A and B, respectively.
Claim: $\pi_A(x) = f\pi_B(x)$ for all $x \in L$: Assume that there exists $x \in L$ such that $(\pi_A - f\pi_B)(x) \neq 0$. Since $A \leq^e E(A)$, there exists $r \in R$ such that $0 \neq (\pi_A - f\pi_B)(xr) \in A$. As $xr \in L$ and $\pi_B(xr) + f\pi_B(xr) \in X \subseteq L$, we have

$$\pi_A(xr) - f\pi_B(xr) = xr - (\pi_B(xr) + f\pi_B(xr)) \in A \cap L = 0,$$

a contradiction. Thus $\pi_A(x) = f\pi_B(x)$ for all $x \in L$.

For $x \in L$, we have

$$x = \pi_A(x) + \pi_B(x) = f(\pi_B(x)) + \pi_B(x) \in X,$$

which implies $L \subseteq X$.

(2) $\Rightarrow$ (3) If the module M has a decomposition $M = A \oplus B$ for an isomorphism $f \in Hom(E(B), E(A))$, we obtain $M = A \oplus X$ with $X = \{b + f(b)|\ b \in B, f(b) \in A\}$. Clearly, $f(B) \leq A$ and hence A is automorphism B-invariant by Lemma 2.2. Symmetrically, $f(A) \leq B$ and so A is automorphism and $A \cong B$.

(3) $\Rightarrow$ (1) This is a direct consequence of Lemma 2.8.

(1) $\Rightarrow$ (4) This follows from Theorem 2.9(2).

(4) $\Rightarrow$ (3) If $E(A) \ncong E(B)$ for each decomposition $M = A \oplus B$, there is nothing to prove. Assume that M has a decomposition $M = X_1 \oplus X_2$ for submodules X_1 and X_2 of M such that $X \cong X_1 \cong X_2$. We suppose furthermore that M has an another decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$. By [3, Theorem 3], both the modules X_1, X_2 and M satisfy the exchange property. Thus there exist submodules $P_1 \subseteq A$, $Q_1 \subseteq B$ such that $M = X_1 \oplus P_1 \oplus Q_1$. Note that $X_2 \cong M/X_1 \cong P_1 \oplus Q_1$ is automorphism invariant, hence there exist submodules $P_2 \subseteq A$, $Q_2 \subseteq B$ such that $M = P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$. Clearly, as $P_1 \oplus P_2 \subseteq A$ and $Q_1 \oplus Q_2 \subseteq B$, we get $A = P_1 \oplus P_2$ and $B = Q_1 \oplus Q_2$, hence

$$E(P_1) \oplus E(P_2) \cong E(P_1 \oplus P_2) \cong E(A)$$

and

$$E(Q_1) \oplus E(Q_2) \cong E(Q_1 \oplus Q_2) \cong E(B).$$

Now, since $E(A) \cong E(B)$, the hypothesis of (4) implies that $A \cong B$ is automorphism invariant. □

For modules M and N, N is said to be M-injective if every homomorphism from each submodule of M to N extends to a homomorphism from M to N, and

M and N are called relatively injective if N is M-injective and M is N-injective. The module M is called quasi-injective if M is M-injective. It is well-known that a module is quasi-injective if and only if it is invariant under automorphisms and idempotent endomorphisms of its injective hull.

In [8], Lee and Zhou discussed when an automorphism invariant module is quasi-injective or injective and they obtained the following observation.

Lemma 2.11 ([8, Theorem 5]). *If $M \oplus N$ is automorphism invariant, then M and N are relatively injective.*

Combining Lemmas 2.8 and 2.11, we have

Corollary 2.12. Every automorphism invariant module is e-ADS.

The following example shows that the converse of Corollary 2.12 is not true in general.

Example 2.13. Take any continuous module M which is not quasi-injective (e.g. if R is the ring of all sequences of real numbers that are eventually rational, then R_R is continuous but not quasi-injective), then clearly M is ADS (and hence e-ADS) but not automorphism invariant.

We recall Example 2.6. It also shows that e-ADS modules are not closed with respect to general direct summands. On the other hand, Corollary 2.12 and Theorem 2.9 prove that the class of all e-ADS modules is closed under taking some important cases of direct summands. We can then show:

Corollary 2.14. Let M be an e-ADS module. If M has a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then A is e-ADS.

In view of the claim of Theorem 2.9, we say that a module M is trivial e-ADS if it has no a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$.

The following observation shows that the trivial e-ADS modules can be described using lattices of their submodules.

Proposition 2.15. Let M be a module. Then M is trivial e-ADS if and only if for every decomposition $M = A \oplus B$ no complement of A is a complement of B.

Proof. Suppose that the module M has a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$. The isomorphism $\varphi : E(B) \cong E(A)$ implies that the restriction of φ on $C = \varphi^{-1}(A) \cap N$ forms an essential monomorphism $\psi : C \to A$. Put $H := \{c - \varphi(c)| : a \in C\}$. Now if we follow the same way as in the proof of (2) $\Rightarrow$ (1) of Lemma 2.1, we have fixed a complement K of B containing H. Since $K \cap B = 0$ and $A \cap (K + B) \leq^e A$, we obtain that K is complement of B.

Conversely, suppose that M has a decomposition $M = A \oplus B$ and K is simultaneously complement of A and B. Then

$$E(M) = E(A) \oplus E(K) = E(B) \oplus E(K),$$

hence $E(A) \cong E(B)$ (here we notice that all injective hulls are considered as submodules of $E(A)$). □

Now, we provide several useful necessary conditions of trivial e-ADS modules.

Lemma 2.16. *Let M be a nonzero module. If every idempotent of $\mathrm{End}(M)$ can be extended to a central idempotent of $\mathrm{End}(E(M))$, then M is trivial e-ADS.*

Proof. Suppose that M has a decomposition $M = A \oplus B$ and consider an idempotent $e \in \mathrm{End}(M)$ defined by the rule $e(a+b) = a$ for all $a \in A$, $b \in B$. By the hypothesis, there exists a central idempotent $\tilde{e} \in \mathrm{End}(E(M))$ satisfying $\tilde{e}(m) = e(m)$ for each $m \in M$. Now, assume that we have an isomorphism $i : E(A) \to E(B)$ and extend it to an endomorphism $j \in \mathrm{End}(E(M))$ such that $j(a + b) = i(a)$ for all $a \in E(A)$ and $b \in E(B)$. Since $A \neq 0 \neq B$ by the hypothesis and i is an isomorphism, $i(A) \cap B$ is essential in $E(B)$, hence there exists nonzero element $a \in A$ for which $0 \neq i(a) \in B$. As $\tilde{e}$ is central, i.e. $\tilde{e}j = j\tilde{e}$, we have

$$0 \neq i(a) = j(a) = je(a) = j\tilde{e}(a) = \tilde{e}j(a) = ei(a) = 0,$$

a contradiction. □

Since every idempotent endomorphism of a module M can be extended to an idempotent endomorphism of $E(M)$ we obtain the following consequence:

Corollary 2.17. If M is a nonzero module such that every idempotent of $\mathrm{End}(E(M))$ is central, then M is trivial e-ADS.

A right R-module M is said to satisfy *the exchange property* if for every right R-module A and any two direct sum decompositions $A = M_1 \oplus N = \oplus_{i \in I} A_i$ with $M_1 \cong M$, there exist submodules B_i of A_i such that $A = M_1 \oplus (\oplus_{i \in I} B_i)$.

A ring R is called *semiregular* if, for every $a \in R$, there exists $b \in R$ such that $bab = b$ and and $a - aba \in J(R)$ ([10]).

Theorem 2.18. *Let M be a non trivial e-ADS module. Then*

(1) *M satisfies the exchange property.*
(2) *$End(M)$ is semiregular.*

Proof. (1) By Theorem 2.9(2), we obtain $M \cong A \oplus A$ where A is automorphism invariant. Moreover, A satisfies the exchange property by [3, Theorem 3]. Hence M satisfies the exchange property because the class of modules satisfying the exchange property is closed under taking finite direct sums.
(2) It follows from Theorem 2.9(2), [3, Proposition 1] and [11, Theorem 29]. □

Recall an easy observation about central idempotents.

Lemma 2.19. *Let A and B be direct summands of a module M and f a central idempotent of* $\mathrm{End}(M)$. *If $A \cong B$, then $f(A) \cong f(B)$.*

Proof. Let $\varphi : A \to B$ be an isomorphism and consider the natural projection $\pi_A : M \to A$ and the natural embedding $\nu_B : B \to M$. Put $h = \nu_B \varphi \pi_A \in \mathrm{End}(M)$. Since f is a central idempotent we get $h = fhf \oplus (1-f)h(1-f)$, hence fhf induces an isomorphisms between $f(A)$ and $f(B)$. □

Note that direct sums of two e-ADS modules need not be e-ADS (as it can be illustrated, e.g. by the direct sum of two trivial e-ADS modules $\mathbb{Z}_2$ and $\mathbb{Z}_4$). The following theorem shows some kind of restrictive closure property of e-ADS modules.

Theorem 2.20. *Let M be a trivial e-ADS and N a nontrivial e-ADS module. If* $\mathrm{Hom}(E(M), E(N)) = 0 = \mathrm{Hom}(E(N), E(M))$, *then $M \oplus N$ is trivial e-ADS.*

Proof. Let $X = M \oplus N$ and assume that there exists a decomposition $X = A \oplus B$ such that $E(A)$ and $E(B)$ are isomorphic. Note that we may suppose all modules and their injective hulls as submodules of $E(X)$.

Since N satisfies exchange property by Theorem 2.18, there exist submodules $C \subseteq A$ and $D \subseteq B$ such that $X = N \oplus C \oplus D$. Obviously, $M \cong X/N \cong C \oplus D$. Thus $E(M) \cong E(C) \oplus E(D)$ where $E(C)$ and $E(D)$ are considered as submodules of $E(A)$ and $E(B)$, respectively. Note that there are injective submodules $E_A \subseteq E(A)$ and $E_B \subseteq E(B)$ for which $E_A \oplus E(C) = E(A)$ and $E_B \oplus E(D) = E(B)$. Now it is easy to see that $E(N) \cong E_A \oplus E_B$. By the hypothesis, we get $\mathrm{End}(E(X)) \cong \mathrm{End}(E(M)) \times \mathrm{End}(E(N))$, hence there exists a central idempotent $f \in \mathrm{End}(E(X))$ for which $f(E(X)) = E(M)$ and $(1-f)(E(X)) = E(N)$. By Lemma 2.19, we obtain that $f(E(A)) \cong f(E(B))$. As $f(E(A)) = E(C)$ and $f(E(B)) = E(D)$, a contradiction. □

3. Classes of e-ADS modules and some ring conditions

Let $\sigma[M]$ denote the Wisbauer category of a module M, i.e. the full category of R-Mod consisting of submodules of quotients of direct sums of copies of M (see [14]).

Theorem 3.1. *The following conditions are equivalent for a module M:*

1. *M is semisimple.*
2. *Every module in $\sigma[M]$ is e-ADS.*
3. *Every finitely generated module in $\sigma[M]$ is e-ADS.*
4. *Every 4-generated module in $\sigma[M]$ is e-ADS.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) are clear.

(4) ⇒ (1) Let $N \in \sigma[M]$ be a cyclic module and $x \in M$. Then

$$(N \oplus xR) \oplus (N \oplus xR)$$

is a 4-generated module in $\sigma[M]$ and hence is e-ADS by the hypothesis. By Lemma 2.8, $N \oplus xR$ is automorphism $N \oplus xR$-invariant and N is xR-injective by Lemma 2.11. By [9, Theorem 1.4], N is M-injective. Thus M is semisimple by [4, Corollary 7.14]. □

Theorem 3.1 gives immediately the following.

Corollary 3.2. A ring R is semisimple Artinian if and only if every 4-generated R-module is e-ADS.

The following observation gives an another characterization of e-ADS modules in the category $\sigma[M]$.

Theorem 3.3. *The following conditions are equivalent for a module M:*

1. *M is semisimple.*
2. *The direct sum of every two e-ADS modules in $\sigma[M]$ is e-ADS.*
3. *Every e-ADS module in $\sigma[M]$ is M-injective.*
4. *The direct sum of any family of e-ADS modules in $\sigma[M]$ is e-ADS.*

Proof. (1) ⇒ (4) ⇒ (2) They are obvious.

(2) ⇒ (3) Let N be an e-ADS module. By our assumption, $(N \oplus E_M(N)) \oplus (N \oplus E_M(N))$ is e-ADS. Then $N \oplus E_M(N)$ is automorphism invariant. Hence N is $E_M(N)$-injective by Lemma 2.11. It follows that N is M-injective.

(3) $\Rightarrow$ (1) We consider a family $\{S_i | i \in \mathbb{N}\}(\subset \sigma[M])$ of simple right R-modules. It follows that $\oplus_{i\in\mathbb{N}} S_i$ is semisimple and so is e-ADS. By (3), $\oplus_{i\in\mathbb{N}} S_i$ is M-injective. Therefore $\oplus_{i\in\mathbb{N}} S_i$ is a direct summand of $\oplus_{i\in\mathbb{N}} E_M(S_i)$. But $\oplus_{i\in\mathbb{N}} S_i$ is essential in $\oplus_{i\in\mathbb{N}} E_M(S_i)$ and then $\oplus_{i\in\mathbb{N}} S_i = \oplus_{i\in\mathbb{N}} E_i$ is M-injective. Thus M is locally Noetherian. We can write $E_M(M) = \oplus_{i\in I} K_i$ for some indecomposable right R-modules K_i in $\sigma[M]$ by [14, 27.4]. We have that every K_i is M-injective and obtain that every K_i is uniform. For each $i \in I$, let $0 \neq x \in K_i$. Since K_i is uniform, xR is uniform as well, hence xR is e-ADS. Then xR is M-injective by (3). It follows that xR is a direct summand of K_i and we have $xR = K_i$. Thus K_i is simple for all $i \in I$. That means $E_M(M)$ is semisimple. Thus M is semisimple. □

Corollary 3.4. The following conditions are equivalent for a ring R:

(1) R is semisimple Artinian.
(2) The direct sum of every two e-ADS modules is e-ADS.
(3) Every e-ADS module is injective.
(4) The direct sum of any family of e-ADS modules is e-ADS.

We note that if $M \oplus E(M)$ is e-ADS for an R-module M, then $M \cong E(M)$ by Theorem 2.9 and so M is injective.

Theorem 3.5. *The following conditions are equivalent for a ring R:*

(1) *R is right Noetherian.*
(2) *The direct sum of injective right R-modules is e-ADS.*
(3) *For any injective right R-module X, $X^{(\mathbb{N})}$ is e-ADS.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) They are obvious.

(3) $\Rightarrow$ (1) Let X be an injective module. Clearly, $X \oplus E(R_R)$ is also injective. Let $M = X \oplus E(R_R)$. Since $4 \cdot |\mathbb{N}| = |\mathbb{N}|$, we obtain that $(M^{(\mathbb{N})})^{(4)} \cong M^{(\mathbb{N})}$. By (3), $M^{(\mathbb{N})} \oplus M^{(\mathbb{N})}$ is automorphism invariant. It follows that $M^{(\mathbb{N})}$ is quasi-injective. On the other hand, $X^{(\mathbb{N})}$ is isomorphic to a direct summand of $M^{(\mathbb{N})}$. It implies that $X^{(\mathbb{N})}$ is $E(R_R)$-injective and so $X^{(\mathbb{N})}$ is injective. Hence R is right Noetherian. □

A ring R is called a right *V-ring* if every simple right R-module is injective.

Theorem 3.6. *The following conditions are equivalent for a ring R:*

(1) *R is a right V-ring,*
(2) *$S \oplus E(S)$ is e-ADS for every simple right R-module S.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Assume that $S \oplus E(S)$ is e-ADS for every simple right R-module S. Let S be a simple right R-module. By the hypothesis, $S \oplus E(S)$ is e-ADS. Then, by Theorem 2.9(1), $S \cong E(S)$, and so S is injective. □

Theorem 3.7. *The following conditions are equivalent for a ring R:*

(1) *R is a QF-ring.*
(2) *Every projective right R-module is e-ADS.*
(3) *Every essential extension of any free right R-module is e-ADS.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1) Let I be a non-empty set. Clearly $(R^{(I)})^4$ is also a projective module. By (2), $R^{(I)} \oplus R^{(I)}$ is automorphism invariant. It follows that $R^{(I)}$ is quasi-injective. Therefore $R^{(I)}$ is injective. Thus R is Σ-injective and so R is a QF-ring.

(3) $\Rightarrow$ (1) Let F be a free right R-module. Then $F \oplus E(F)$ is an essential extension of a free right module F^2. By (3), $F \oplus E(F)$ is e-ADS, hence F is injective. Now we have proved that every projective right R-module is injective. Thus R is QF by the Faith-Walker theorem. □

4. The Structure of e-ADS rings

We say that a ring R is right e-ADS if it is an e-ADS module over itself. A right e-ADS ring R is called trivial if R_R is trivial e-ADS, i.e. the module R_R does not have a decomposition $R_R = A \oplus B$ such that $E(A) \cong E(B)$. Otherwise R is said to be a nontrivial e-ADS ring.

Let R be a ring, e be an idempotent of R, $S := eRe$ and $n \in \mathbb{N}$. Denote by $\mathcal{L}(eR^n)$ the lattice of all submodules of the projective R-module eR^n, and $\mathcal{L}(S^n)$ the lattice of all submodules of the free module S^n. Define two mappings

$$\Phi : \mathcal{L}(eR^n) \to \mathcal{L}(S^n)$$

and

$$\Psi : \mathcal{L}(S^n) \to \mathcal{L}(eR^n)$$

by the rules

$$\Phi(I) = Ie, \qquad \Psi(J) = JR$$

for arbitrary $I \in \mathcal{L}(eR^n)$ and $J \in \mathcal{L}(S^n)$.

Lemma 4.1. *Φ and Ψ are well-defined monotonic mappings. Moreover, Φ is a lattice homomorphism and Ψ is compatible with the operation $+$.*

Proof. Straightforward from the above notation. □

Note that the inclusion $\Psi(J_1 \cap J_2) \subseteq \Psi(J_1) \cap \Psi(J_2)$ holds generally for arbitrary $J_1, J_2 \in \mathcal{L}(S^n)$ but the following example shows that the reverse need not be true.

Example 4.2. Let $R = \{(a_{ij}) \in M_{3\times 3}(\mathbb{Q})|\ a_{31} = a_{32} = 0\}$ be a subring of matrix ring $M_{3\times 3}(\mathbb{Q})$. Put $e := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $g := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $S := eRe$, $J_1 := fS$, and $J_2 := gS$. Then it is easy to see that

$$J_1 \cap J_2 = 0$$

and

$$J_1R \cap J_2R = \{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \mid u, v \in \mathbb{Q}\}.$$

Thus $(J_1 \cap J_2)R \neq J_1R \cap J_2R$.

Lemma 4.3. *Let R be a ring and $e \in R$ be an idempotent such that $ReR = R$. Then Φ and Ψ are mutually inverse lattice isomorphisms.*

Proof. Let $S := ReR$. Since both Φ and Ψ are monotonic, it is enough to show that $\Phi\Psi$ and $\Psi\Phi$ are identity mappings on $\mathcal{L}(S)$ and $\mathcal{L}(eR)$, respectively. Let $I \in \mathcal{L}(eR)$ and $J \in \mathcal{L}(S)$. Since $ReR = R$, we get

$$\Psi\Phi(I) = IeR = IReR = IR = I.$$

On the other hand $S = eRe$ and $J = Je$ imply that

$$\Phi\Psi(J) = JRe = JeRe = JS = J.$$

□

Recall that essentiality of modules can be expressed as a condition of lattices of submodules:

Lemma 4.4. *Let $A \subseteq B$ are submodules of a module M. Then $A \leq^e B$ if and only if there exists no submodule $C \subseteq B$ such that $A \cap C = 0$.*

Proof. This is well known. □

The following general consequence is a special case of [15, Theorem 1.2] for the lattice isomorphism from Lemma 4.3.

Corollary 4.5. Let R and S be rings, M an R-module, N an S-module and K, L submodules of M. Suppose that $\phi : \mathcal{L}(M_R) \to \mathcal{L}(N_S)$ is an isomorphism of lattices of all submodules of M and N. Then K is a complement of L if and only if $\phi(K)$ is a complement of $\phi(L)$.

Lemmas 4.4, 2.1, 2.15 and Corollary 4.5 show that e-ADS, trivial e-ADS and relative automorphism invariant are lattice conditions. Thus the assertions of the following theorem hold true because lattices of all submodules of M and N are isomorphic.

Theorem 4.6. *Let R and S be rings, M an R-module and N an S-module. Assume $\phi : \mathcal{L}(M_R) \to \mathcal{L}(N_S)$ is an isomorphism of lattices.*

1. *M is (trivial) e-ADS if and only if N is a (trivial) e-ADS.*
2. *If $M = A \oplus B$, then $N = \phi(A) \oplus \phi(B)$ and A is B-automorphism invariant if and only if $\phi(A)$ is $\phi(B)$-automorphism invariant.*

Let $n \in \mathbb{N}$ and e be an idempotent of a ring R such that $ReR = R$. Recall that $L(eR^n_R)$ and $L(S^n_S)$ are isomorphic lattices by Lemma 4.3 for every $n \in \mathbb{N}$, where $S = eRe$.

Theorem 4.7. *Let R be a ring, $n \in \mathbb{N}$ and $e \in R$ be an idempotent such that $ReR = R$.*

1. *eR^n_R is a (trivial) e-ADS module if and only if $eR^n e$ is (trivial) e-ADS as a right eRe-module.*
2. *Let $eR^n = A \oplus B$. Then A is B-automorphism invariant if and only if Ae is Be-automorphism invariant.*
3. *eR is automorphism invariant if and only if S_S is automorphism invariant, where $S = eRe$.*

Proof. (1) and (2) follow immediately from Theorem 4.6.
(3) It suffices to apply (2) for the decomposition $eR^2 = eR \oplus eR$. □

The next observation shows that the class of e-ADS rings is closed under taking finite products.

Proposition 4.8. If R_1 and R_2 are e-ADS rings, then $R_1 \times R_2$ is e-ADS as well.

Proof. Put $R := R_1 \times R_2$ and let e_i be orthogonal central idempotents such that $R_i = Re_i$ for $i = 1, 2$. It is easy to see that $e_1 + e_2 = 1$, $E(R) = E(R_1) \oplus E(R_2)$ and $E(R_i) = E(R)e_i$ for $i = 1, 2$. Suppose that $R = A \oplus B$ is a module decomposition, $C \leq^e A$, $D \leq^e B$ and $f : C \to D$ is an isomorphism. Then $f_i : Ce_i \to De_i$ defined by $f_i(r) = re_i$ is an isomorphism for each $i = 1, 2$. We note that $Ce_i \leq^e Ae_i$ and

$De_i \leq^e Be_i$ for each $i = 1, 2$. By the hypothesis, there exist extensions $g_i : Ae_i \to Be_i$ of f_i. Clearly, $g = g_1 \oplus g_2 : A \to B$ extends f. □

We denote the set of all $n \times n$ matrices over a ring R by $M_n(R)$.

Lemma 4.9. *If R is a non-trivial e-ADS ring, then there exists a right automorphism invariant ring S such that $R \cong M_2(S)$.*

Proof. Since R is a non-trivial e-ADS ring, there exists an idempotent $e \in R$ for which $E(eR) \cong E((1-e)R)$. Thus $eR \cong (1-e)R$ is automorphism invariant by Theorem 2.9. Put $S := eRe$. Then

$$R \cong \text{End}(eR \oplus eR) \cong M_2(S)$$

and S is a right automorphism invariant ring by Theorem 4.7(3). □

Let R be a ring. Recall that R is said to be right non-singular if its right singular ideal $Z(R) = \{r \in R : rI = 0 \text{ for some essential right ideal } I \text{ of } R\}$ is zero, and R is called normal if if moreover its idempotents are central. Note that every abelian regular ring or every product of rings without non-trivial idempotents can serve as elementary examples of normal rings.

Proposition 4.10. Let R be a right non-singular normal automorphism invariant ring. Then

(1) R is trivial e-ADS,
(2) $M_2(R)$ is non-trivial e-ADS.

Proof. Denote by Q the maximal right ring of quotients R. Obviously $eQ = E(eR)$ for every idempotent e.

(1) As every central idempotent of R is a central idempotent of Q, the assertion follows from Lemma 2.16.

(2) By Theorem 4.7 it is enough to prove that $M = R \oplus R$ is a non-trivial e-ADS module. Clearly, M cannot be trivial. So it suffices to prove Theorem 2.10(4). Suppose $R = e_iR \oplus f_iR$ for every $i = 1, 2$, where (e_i, f_i) is a pair of orthogonal idempotents such that $e_1Q \oplus e_2Q \cong f_1Q \oplus f_2Q$. We claim that $A := e_1R \oplus e_2R \cong B := f_1R \oplus f_2R$ (and that A is automorphism invariant).

Since R is a normal ring, i.e., all idempotents e_i, f_i of R, are central for each $i = 1, 2$, we have

$$e_iQ = e_ie_jQ \oplus e_if_jQ$$
$$f_iQ = f_ie_jQ \oplus f_if_jQ$$

for $i \neq j$. Hence $Q = e_1e_2Q \times e_1f_2Q \times f_1e_2Q \times f_1f_2Q$, where there is no nonzero homomorphism between two distinct components. Thus

$$E(A) = e_1Q + e_2Q \cong (e_1e_2Q)^{(2)} \oplus e_1f_2Q \oplus e_2f_1Q$$

and

$$E(B) = f_1Q + f_2Q \cong (f_1f_2Q)^{(2)} \oplus e_1f_2Q \oplus e_2f_1Q.$$

We have observed that $\text{Hom}(e_1e_2Q, E(B)) = 0$ as well as $\text{Hom}(e_1e_2Q, E(B)) = 0$ which implies that $e_1e_2 = 0 = f_1f_2$. Hence

$$E(A) \cong e_1f_2Q \oplus e_2f_1Q \cong E(B)$$

and so

$$A \cong e_1f_2R \oplus e_2f_1R \cong B.$$

Finally, since $e_1f_2R \oplus e_2f_1R$ is isomorphic to a direct summand of R which is automorphism invariant, we obtain that A is automorphism invariant by [8, Lemma 4]. □

We finish the section with the following criterion.

Theorem 4.11. *Let R be a right non-singular ring and Q be its the maximal right ring of quotients. Then the following is equivalent:*

(1) *R is right e-ADS,*
(2) *Either $eQ \ncong (1-e)Q$ for any idempotent $e \in R$ or $R \cong M_2(S)$ for a suitable right automorphism invariant ring S,*
(3) *Either $eQ \ncong (1-e)Q$ for any idempotent $e \in R$ or $R \cong T \times M_2(S)$ for a suitable self-injective ring T and a normal right automorphism invariant ring S.*

Proof. (1) $\Rightarrow$ (2) If R is a right trivial e-ADS ring, then $Q \cong E(R)$ has no a decomposition $Q = A \oplus B$ with a isomorphic summand, which implies that $eQ \ncong (1-e)Q$ for any idempotent $e \in R$.

If R is a non-trivial e-ADS ring, then there exists a right automorphism invariant ring S such that $R \cong M_2(S)$ by Lemma 4.9.

(2) $\Rightarrow$ (3) Assume $R \cong M_2(S_0)$ for a right automorphism invariant ring S_0. Clearly, S_0 is, moreover, non-singular, hence there exists a right selfinjective ring S_1 and a normal right automorphism invariant ring S such that $S_0 \cong S_1 \times S$ by [5, Theorem 7]. Now it is easy to see that

$$M_2(S_0) \cong M_2(S_1) \times M_2(S)$$

and $T = M_2(S_0)$ is self-injective by [7, Corollary 9.3].

(3) $\Rightarrow$ (1) We remark that the first condition implies that R is a trivial e-ADS ring. Suppose that $R \cong T \times M_2(S)$ where T is a self-injective ring and S is a normal right automorphism invariant ring. Note that T is an e-ADS ring and $M_2(S)$ is e-ADS by Lemma 4.10. So, R is right e-ADS by Lemma 4.8. □

Corollary 4.12. Every simple non-trivial right e-ADS ring is necessarily self-injective.

Proof. It follows from Theorem 4.11 and [5, Corollary 10]. □

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