ESSENTIALLY ADS MODULES AND RINGS

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Abstract. This paper introduces the notion of essentially ADS (e-ADS) modules. Basic structural properties and examples of e-ADS modules are presented. In particular, it is proved that (1) The class of all e-ADS modules properly contains all ADS as well as automorphism invariant modules, e-ADS modules serve also as a tool for characterization of various classes of rings. It is shown that: (2) R is a QF-ring if and only if every projective right R-module is e-ADS; (3) R is a semisimple Artinian ring if and only if every e-ADS module is injective. The final part of this paper describes properties of e-ADS rings, which allow to prove a criterion of e-ADS modules for non-singular rings: (4) Let R be a right non-singular ring and Q be its the right maximal ring of quotients. Then R is a right e-ADS ring if and only if either \( eQ \neq (1-e)Q \) for any idempotent \( e \in R \) or \( R \cong M_2(A) \) for a suitable right automorphism invariant ring A.

1. Introduction

The absolute direct summand (ADS) property for modules was introduced by Fuchs in [6] and recently was intensively studied by Alahmadi, Jain and Leroy in [1]. Recall that a right module \( M \) over a ring \( R \) is said to be ADS if for every decomposition \( M = S \oplus T \) and every complement \( T' \) of \( S \), we have \( M = S \oplus T' \).

In recent works [5, 8, 12], the notion of automorphism invariant modules was shown to be an important tool for finding correspondences between various concepts of injectivity. A module \( M \) is called automorphism invariant if it is invariant under automorphisms of its injective hull, equivalently if every isomorphism between two essential submodules of \( M \) extends to an automorphism of \( M \) [8]. Quasi-injective modules are automorphism invariant. Assume that an \( R \)-module \( M \) has a decomposition \( M = S \oplus T \) such that \( T' \) is a complement of \( S \), \( T' \cap T = 0 \) and \( S \cap (T' \oplus T) \leq_S S \). It is easy to see (cf. Lemma 2.4) that \( E(S) \cong E(T) \), where \( E \) denotes the injective hull. In light of this observation, we define essentially ADS-modules (shortly e-ADS), as an \( R \)-module \( M \) such that for every decomposition \( M = S \oplus T \) of \( M \) and every complement \( T' \) of \( S \) with \( T' \cap T = 0 \) and \( S \cap (T' \oplus T) \leq_S S \), we have \( M = S \oplus T' \). This definition naturally generalizes both notions mentioned above. Furthermore, recall that when a module \( M \) is quasi-continuous, for each decomposition \( M = A \oplus B \), \( A \) and \( B \) are relatively injective. This property of modules is known to be equivalent to ADS modules ([1, Lemma 3.1]) and automorphism invariant modules ([8, Theorem 3]). Since an \( R \)-module \( M \) is e-ADS if and only if for each decomposition \( M = A \oplus B \), \( A \) and \( B \) are relatively automorphism invariant (see Lemma 2.8), e-ADS modules arise as a

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generalization of quasi-continuous modules, ADS modules as well as automorphism invariant modules. The goal of this paper is to present a list of significant structural properties of e-ADS modules and to exhibit relations with other notions. We show that e-ADS as well as automorphism invariant or ADS modules have a description in the language of the lattice theory (Lemmas 2.1, 2.15, 4.4). Important for further study is the division of the class of e-ADS modules into trivial and non-trivial case (cf. Theorem 2.9). Moreover, it is proved in Theorem 2.9 that, if \( E(A) \not= E(B) \) for each decomposition \( M = A \oplus B \), then \( M \) is e-ADS. On the other hand, if \( M \) is an e-ADS module with a decomposition \( M = A \oplus B \) such that \( E(A) \equiv E(B) \), then \( A \equiv B \) and the modules \( A \) and \( B \) are automorphism invariant. This result is key to our work and is used to characterize many well-known classes of modules in terms of e-ADS modules. For example, we show in Theorem 2.18 that for an e-ADS module \( M \) with a decomposition \( M = A \oplus B \) such that \( E(A) \equiv E(B) \), \( M \) satisfies the exchange property if and only if \( End(M) \) is semiregular.

The final part of the article is devoted to rings which are e-ADS as right modules over themselves. By applying elementary lattice theoretical tools on rings induced by idempotents we characterize when non-singular rings are e-ADS. Based on the key observation that a non-trivial e-ADS ring is isomorphic to a \( 2 \times 2 \) full matrix ring over an automorphism invariant ring (Lemma 4.9) we prove a characterization of non-singular e-ADS rings. Namely, a non-singular ring is e-ADS if and only if it is either trivial e-ADS or it is a product of a self-injective ring and a matrix ring \( M_2(S) \) over an automorphism invariant ring \( S \) with many central idempotents (Theorem 4.11).

Throughout this article, unless otherwise stated, all rings have unity and all modules are unital. For a submodule \( N \) of \( M \), we use \( N \leq M (N < M) \) to mean that \( N \) is a submodule of \( M \) (respectively, proper submodule), and we write \( N \leq^e M \) to indicate that \( N \) is an essential submodule of \( M \). For any term not defined here the reader is referred to [2], [4] and [9].

2. e-ADS modules

Let \( M \) and \( N \) be two modules. The module \( M \) is called automorphism \( N \)-invariant if for any essential submodule \( A \) of \( N \), any essential monomorphism from \( A \) to \( M \) can be extended to some \( g \in \text{Hom}(N, M) \) ([12]).

We note that \( M \) is automorphism invariant if \( M \) is automorphism \( M \)-invariant by [8, Theorem 2].

**Lemma 2.1.** Let \( M \) and \( N \) be modules and \( X = M \oplus N \). The following conditions are equivalent:

1. \( M \) is automorphism \( N \)-invariant.
2. For any complement \( K \) of \( M \) in \( X \) with \( K \cap N = 0 \) and \( M \cap (K \oplus N) \leq^e M \), the module \( X \) has a decomposition \( X = M \oplus K \).

**Proof.** Consider the natural projections \( \pi_M : X \to M \) and \( \pi_N : X \to N \). Note that \( \pi_M(K) = M \cap (K + N) \) for each submodule \( K \) of \( X \).

(1) \( \Rightarrow \) (2) Let \( K \) be a complement of \( M \) in \( X \) with \( K \cap N = 0 \) and \( \pi_M(K) \leq^e M \). Clearly, \( M \oplus K = M \oplus \pi_N(K) \) so that \( \pi_N(K) \) is essential in \( N \). Consider the homomorphism \( \theta : \pi_N(K) \to \pi_M(K) \) defined by \( \theta(n) = m \) whenever \( k = m + n \in K \) for \( k \in K, m \in M, n \in N \). It is easy to see that \( \theta \) is an isomorphism \( (K \cap N = 0 \) by the assumption). Since \( M \) is automorphism \( N \)-invariant, the homomorphism \( \theta \) can
be extended to some \( g : N \to M \). Set \( T := \{ n + g(n) \mid n \in N \} \). Clearly, \( M \oplus T = X \) and \( T \) contains \( K \) essentially by modularity. Since \( K \) is a complement, we obtain \( T = K \).

\((2) \Rightarrow (1)\) Let \( A \) be an essential submodule of \( N \) and \( f : A \to M \) be an essential monomorphism. Set \( H := \{ a - f(a) \mid a \in A \} \). Clearly, \( H \cap N = 0 \), \( H \cap M = 0 \) and \( \pi_M(H) = f(A) \) is essential in \( M \). Then \( M \oplus H = M \oplus \pi_N(H) = M \oplus A \), which is essential in \( X \). Let \( K \) be a complement of \( M \) in \( X \) containing \( H \). Then \( H \leq^e K \). Hence \( K \cap N = 0 \) because \( H \cap N = 0 \). Moreover, \( \pi_M(H) \leq \pi_M(K) \) which implies that \( \pi_M(K) \leq^e M \). By the assumption, we have \( M \oplus K = X \). Now let \( \pi : M \oplus K \to M \) be the projection. Then writing an element \( a \in A \) in the form \( a = a - f(a) + f(a) \), the restriction of \( \pi \) to \( N \) is the desired extension of \( f \).

**Lemma 2.2** ([12, Theorem 2.2]). The following are equivalent for modules \( M \) and \( N \):

1. \( M \) is automorphism \( N \)-invariant.
2. \( \alpha(N) \leq M \) for every isomorphism \( \alpha : E(N) \to E(M) \).

As an immediate consequence of Lemmas 2.1 and 2.2, we obtain the following observation.

**Corollary 2.3.** If \( M \) and \( N \) are relatively automorphism invariant modules and \( E(M) \equiv E(N) \), then \( M \equiv N \).

**Lemma 2.4.** Let \( M \) be a module with a decomposition \( M = S \oplus T \). If \( T' \) is a complement of \( S \) with \( T' \cap T = 0 \) and \( S \cap (T' \oplus T) \leq^e S \), then \( T \oplus T' \leq^e M \) and \( E(S) \equiv E(T) \).

**Proof.** Note that \( S \oplus T' \leq^e M \) because \( T' \) is a complement of \( S \). Since

\[
T \oplus [S \cap (T' \oplus T)] \leq T \oplus T' \quad \text{and} \quad T \oplus [S \cap (T' \oplus T)] \leq^e T \oplus S = M,
\]

we get \( T \oplus T' \leq^e M \). Moreover, the injective hulls \( E(S) \), \( E(T) \) and \( E(T') \) can be taken as submodules of the injective hull \( E(M) \) such that \( S \leq^e E(S) \), \( T \leq^e E(T) \) and \( T' \leq^e E(T') \). Since \( S \cap T' = 0 = T \cap T' \), it is easy to see that

\[
E(S) \cap E(T') = 0 = E(T) \cap E(T').
\]

On the other hand,

\[
E(S) + E(T') = E(M) = E(T) + E(T')
\]

because both \( E(S) + E(T') = E(S) \oplus E(T') \) and \( E(T) + E(T') = E(T) \oplus E(T') \) are injective submodules of \( E(M) \), and both \( S \oplus T' \) and \( T \oplus T' \) are essential in \( E(M) \). Thus

\[
E(T) \equiv (E(T) + E(T'))/E(T') = E(M)/E(T') = (E(S) + E(T'))/E(T') \equiv E(S).
\]

In light of Lemma 2.4, we call \( M \) an essentially ADS-module, shortly e-ADS, if for every decomposition \( M = S \oplus T \) of \( M \) and every complement \( T' \) of \( S \) with \( T' \cap T = 0 \) and \( S \cap (T' \oplus T) \leq^e S \), we have \( M = S \oplus T' \).
Clearly, ADS-modules are e-ADS. The following examples show that the converse is not true in general and that the class of e-ADS modules is not closed under taking direct summands, respectively.

**Example 2.5.** Let $T$ be a torsion abelian group which is not divisible and put $M := \mathbb{Z} \oplus T$. Then every decomposition $M = A \oplus B$ contains a subgroup which is isomorphic to $\mathbb{Z}$ while the second is torsion, hence $E(A) \neq E(B)$. By Lemma 2.4, there exists no a decomposition satisfying the hypothesis of the definition of e-ADS. So that the conditions for having an e-ADS modules are vacuously satisfied and the module is e-ADS. Hence it is an e-ADS abelian group.

On the other hand, since $T$ is not divisible, we obtain $T$ is not $\mathbb{Z}$-injective and so $M$ is not ADS by [1, Lemma 3.1].

**Example 2.6.** Put $M := \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p^2$ for some prime number $p$. Then $M$ is e-ADS by Example 2.5. Since $\mathbb{Z}_p$ is not automorphism $\mathbb{Z}_p^2$-invariant, we obtain that $\mathbb{Z}_p \oplus \mathbb{Z}_p^2$ is not e-ADS by Lemma 2.8.

Let us mention the following equivalent conditions for a module to be e-ADS.

**Theorem 2.7.** The following conditions are equivalent for a module $M$:

1. $M$ is e-ADS.
2. For every decomposition $M = S \oplus T$, if $T'$ is a complement of $S$ in $M$ and $T$ is a complement of $T'$ in $M$, then $M = S \oplus T'$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $M = S \oplus T$ is a decomposition of $M$, $T'$ is complement of $S$ and $T$ is a complement of $T'$ in $M$. Then $S \cap (T' \oplus T) \leq^e S$ since $T \oplus T' \leq^e M$. By (1), we have $M = S \oplus T'$.

(2) $\Rightarrow$ (1) Let $M = S \oplus T$ of $M$ and $S \cap (T' \oplus T) \leq^e S$ for a complement $T'$ of $S$ with $T' \cap T = 0$. By Lemma 2.4, $T \oplus T' \leq^e M$. Since $T$ is a direct summand of $M$, we get $T$ is a complement of $T'$ in $M$. By (2), we have $M = S \oplus T'$.

In [1, Lemma 3.1], it is shown that an $R$-module $M$ is ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are mutually injective.

**Lemma 2.8.** An $R$-module $M$ is e-ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively automorphism invariant.

**Proof.** This is clear from Lemma 2.1. \hfill \Box

The following characterization proves to be quite useful.

**Theorem 2.9.** Let $M$ be an $R$-module.

1. If $E(A) \neq E(B)$ for each decomposition $M = A \oplus B$, then $M$ is e-ADS.
2. If $M$ is an e-ADS module with a decomposition $M = A \oplus B$ such that $E(A) \equiv E(B)$, then $A \equiv B$ and the modules $A$ and $B$ are automorphism invariant.

**Proof.** (1) This follows from Lemmas 2.2 and 2.8.

(2) By Lemma 2.8 and Corollary 2.3, we have $A \equiv B$. Thus $A$ is automorphism $A$-invariant, i.e. automorphism invariant. \hfill \Box

In the following observation, we continue to obtain equivalent conditions for a module to be e-ADS.
Theorem 2.10. The following conditions are equivalent for a module $M$:

1. $M$ is $e$-ADS,
2. Assume that $M$ has a decomposition $M = A \oplus B$. For any isomorphism $f \in \text{Hom}(E(B), E(A))$, the module $M$ has a decomposition $M = A \oplus X$, where $X = \{ b + f(b) | b \in B, f(b) \in A \}$.
3. For every decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, the module $A \cong B$ is automorphism invariant.
4. Either $E(A) \not\cong E(B)$ for every decomposition $M = A \oplus B$ or there exists an automorphism-invariant module $X$ for which $M \cong X \oplus X$ and for every two decompositions $X = P_1 \oplus Q_1 = P_2 \oplus Q_2$ with $E(P_1) \oplus E(P_2) \cong E(Q_1) \oplus E(Q_2)$ we have $(P_1 \oplus P_2) \cong (Q_1 \oplus Q_2)$ is automorphism invariant.

Proof. (1) $\Rightarrow$ (2) We show that $X = \{ b + f(b) | b \in B, f(b) \in A \}$ is a complement of $A$ in $M$. Notice that $A \cap X = 0$, $X \cap B = 0$ and $A \cap (X \oplus B) \leq^e A$. Let $L$ be a submodule of $M$ such that $L \cap A = 0$ and $X \subseteq L$. Consider the natural projections $\pi_A$ and $\pi_B$ of $M$ onto $A$ and $B$, respectively.

Claim: $\pi_A(x) = f\pi_B(x)$ for all $x \in L$. Assume that there exists $x \in L$ such that $(\pi_A - f\pi_B)(x) \neq 0$. Since $A \leq^e E(A)$, there exists $r \in B$ such that $0 \neq (\pi_A - f\pi_B)(xr) \in A$. As $xr \in L$ and $\pi_B(xr) + f\pi_B(xr) \in X \subseteq L$, we have $\pi_A(xr) = f\pi_B(xr) = xr - (\pi_B(xr) + f\pi_B(xr)) \in A \cap L = 0$.

Thus, the natural projections $\pi_A$ and $\pi_B$ of $M$ onto $A$ and $B$, respectively.

For $x \in L$, we have $x = \pi_A(x) + \pi_B(x) = f(\pi_B(x)) + \pi_B(x) \in X$, which implies $L \subseteq X$.

(2) $\Rightarrow$ (3) If the module $M$ has a decomposition $M = A \oplus B$ for an isomorphism $f \in \text{Hom}(E(B), E(A))$, we obtain $M = A \oplus X$ with $X = \{ b + f(b) | b \in B, f(b) \in A \}$. Clearly, $f(B) \leq A$ and hence $A$ is automorphism and $B$-invariant by Lemma 2.2.

Symmetrically, $f(A) \leq B$ and so $A$ is automorphism and $B \cong B$.

(3) $\Rightarrow$ (1) This is a direct consequence of Lemma 2.8.

(1) $\Rightarrow$ (4) This follows from Theorem 2.9(2).

(4) $\Rightarrow$ (3) If $E(A) \not\cong E(B)$ for each decomposition $M = A \oplus B$, there is nothing to prove. Assume that $M$ has a decomposition $M = X_1 \oplus X_2$ for submodules $X_1$ and $X_2$ of $M$ such that $X \cong X_1 \cong X_2$. We suppose furthermore that $M$ has another decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$. By [3, Theorem 3], both the modules $X_1$, $X_2$ and $M$ satisfy the exchange property. Thus there exist submodules $P_1 \subseteq A$, $Q_1 \subseteq B$ such that $M = X_1 \oplus P_1 \oplus Q_1$. Note that $X_2 \cong M/X_1 \cong P_1 \oplus Q_1$ is automorphism invariant, hence there exist submodules $P_2 \subseteq A$, $Q_2 \subseteq B$ such that $M = P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$. Clearly, as $P_1 \oplus P_2 \subseteq A$ and $Q_1 \oplus Q_2 \subseteq B$, we get $A = P_1 \oplus P_2$ and $B = Q_1 \oplus Q_2$, hence $E(P_1) \oplus E(P_2) \cong E(P_1 \oplus P_2) \cong E(A)$ and $E(Q_1) \oplus E(Q_2) \cong E(Q_1 \oplus Q_2) \cong E(B)$.

Now, since $E(A) \cong E(B)$, the hypothesis of (4) implies that $A \cong B$ is automorphism invariant.

For modules $M$ and $N$, $N$ is said to be $M$-injective if every homomorphism from each submodule of $M$ to $N$ extends to a homomorphism from $M$ to $N$, and
and \( N \) are called relatively injective if \( N \) is \( M \)-injective and \( M \) is \( N \)-injective. The module \( M \) is called quasi-injective if \( M \) is \( M \)-injective. It is well-known that a module is quasi-injective if and only if it is invariant under automorphisms and idempotent endomorphisms of its injective hull.

In [8], Lee and Zhou discussed when an automorphism invariant module is quasi-injective or injective and they obtained the following observation.

**Lemma 2.11** ([8, Theorem 5]). *If \( M \oplus N \) is automorphism invariant, then \( M \) and \( N \) are relatively injective.*

Combining Lemmas 2.8 and 2.11, we have

**Corollary 2.12.** Every automorphism invariant module is e-ADS.

The following example shows that the converse of Corollary 2.12 is not true in general.

**Example 2.13.** Take any continuous module \( M \) which is not quasi-injective (e.g. if \( R \) is the ring of all sequences of real numbers that are eventually rational, then \( R_R \) is continuous but not quasi-injective), then clearly \( M \) is ADS (and hence e-ADS) but not automorphism invariant.

We recall Example 2.6. It also shows that e-ADS modules are not closed with respect to general direct summands. On the other hand, Corollary 2.12 and Theorem 2.9 prove that the class of all e-ADS modules is closed under taking some important cases of direct summands. We can then show:

**Corollary 2.14.** Let \( M \) be an e-ADS module. If \( M \) has a decomposition \( M = A \oplus B \) such that \( E(A) \equiv E(B) \), then \( A \) is e-ADS.

In view of the claim of Theorem 2.9, we say that a module \( M \) is trivial e-ADS if it has no a decomposition \( M = A \oplus B \) such that \( E(A) \equiv E(B) \).

The following observation shows that the trivial e-ADS modules can be described using lattices of their submodules.

**Proposition 2.15.** Let \( M \) be a module. Then \( M \) is trivial e-ADS if and only if for every decomposition \( M = A \oplus B \) no complement of \( A \) is a complement of \( B \).

**Proof.** Suppose that the module \( M \) has a decomposition \( M = A \oplus B \) such that \( E(A) \equiv E(B) \). The isomorphism \( \varphi : E(B) \equiv E(A) \) implies that the restriction of \( \varphi \) on \( C = \varphi^{-1}(A) \cap N \) forms an essential monomorphism \( \psi : C \to A \). Put \( H := \{ c - \varphi(c) : a \in C \} \). Now if we follow the same way as in the proof of (2) \( \Rightarrow \) (1) of Lemma 2.1, we have fixed a complement \( K \) of \( B \) containing \( H \). Since \( K \cap B = 0 \) and \( A \cap (K + B) \leq \subseteq A \), we obtain that \( K \) is complement of \( B \).

Conversely, suppose that \( M \) has a decomposition \( M = A \oplus B \) and \( K \) is simultaneously complement of \( A \) and \( B \). Then
\[
E(M) = E(A) \oplus E(K) = E(B) \oplus E(K),
\]
hence \( E(A) \equiv E(B) \) (here we notice that all injective hulls are considered as submodules of \( E(A) \)). \( \square \)

Now, we provide several useful necessary conditions of trivial e-ADS modules.

**Lemma 2.16.** Let \( M \) be a nonzero module. If every idempotent of \( \text{End}(M) \) can be extended to a central idempotent of \( \text{End}(E(M)) \), then \( M \) is trivial e-ADS.
Proof. Suppose that $M$ has a decomposition $M = A \oplus B$ and consider an idempotent $e \in \text{End}(M)$ defined by the rule $e(a + b) = a$ for all $a \in A, b \in B$. By the hypothesis, there exists a central idempotent $e \in \text{End}(E(M))$ satisfying $e(m) = e(m)$ for each $m \in M$. Now, assume that we have an isomorphism $i : E(A) \to E(B)$ and extend it to an endomorphism $j \in \text{End}(E(M))$ such that $j(a + b) = i(a)$ for all $a \in E(A)$ and $b \in E(B)$. Since $A \neq 0 \neq B$ by the hypothesis and $i$ is an isomorphism, $i(A) \cap B$ is essential in $E(B)$, hence there exists nonzero element $a \in A$ for which $0 \neq i(a) \in B$. As $e$ is central, i.e. $\hat{e}j = j\hat{e}$, we have
\[0 \neq i(a) = j(a) = je(a) = j\hat{e}(a) = \hat{e}j(a) = ei(a) = 0,\]
a contradiction. \qed

Since every idempotent endomorphism of a module $M$ can be extended to an idempotent endomorphism of $E(M)$ we obtain the following consequence:

Corollary 2.17. If $M$ is a nonzero module such that every idempotent of $\text{End}(E(M))$ is central, then $M$ is trivial $e$-ADS.

A right $R$-module $M$ is said to satisfy the exchange property if for every right $R$-module $A$ and any two direct sum decompositions $A = M_1 \oplus N = \oplus_{i \in I} A_i$ with $M_1 \cong M$, there exist submodules $B_i$ of $A_i$ such that $A = M_1 \oplus \oplus_{i \in I} B_i$.

A ring $R$ is called semiregular if, for every $a \in R$, there exists $b \in R$ such that $bab = b$ and and $a - aba \in J(R)$ ([10]).

Theorem 2.18. Let $M$ be a non trivial $e$-ADS module. Then

1. $M$ satisfies the exchange property.
2. $\text{End}(M)$ is semiregular.

Proof. (1) By Theorem 2.9(2), we obtain $M \cong \oplus \oplus A$ where $A$ is automorphism invariant. Moreover, $A$ satisfies the exchange property by [3, Theorem 3]. Hence $M$ satisfies the exchange property because the class of modules satisfying the exchange property is closed under taking finite direct sums.

(2) It follows from Theorem 2.9(2), [3, Proposition 1] and [11, Theorem 29]. \qed

Recall an easy observation about central idempotents.

Lemma 2.19. Let $A$ and $B$ be direct summands of a module $M$ and $f$ a central idempotent of $\text{End}(M)$. If $A \cong B$, then $f(A) \cong f(B)$.

Proof. Let $\varphi : A \to B$ be an isomorphism and consider the natural projection $\pi_A : M \to A$ and the natural embedding $\nu_B : B \to M$. Put $h = \nu_B \varphi \pi_A \in \text{End}(M)$. Since $f$ is a central idempotent we get $h = fhf \oplus (1 - f)h(1 - f)$, hence $fhf$ induces an isomorphisms between $f(A)$ and $f(B)$. \qed

Note that direct sums of two e-ADS modules need not be e-ADS (as it can be illustrated, e.g. by the direct sum of two trivial e-ADS modules $\mathbb{Z}_2$ and $\mathbb{Z}_4$). The following theorem shows some kind of restrictive closure property of e-ADS modules.

Theorem 2.20. Let $M$ be a trivial $e$-ADS and $N$ a nontrivial $e$-ADS module. If $\text{Hom}(E(M), E(N)) = 0 = \text{Hom}(E(N), E(M))$, then $M \oplus N$ is trivial $e$-ADS.
Proof. Let \( X = M \oplus N \) and assume that there exists a decomposition \( X = A \oplus B \) such that \( E(A) \) and \( E(B) \) are isomorphic. Note that we may suppose all modules and their injective hulls as submodules of \( E(X) \).

Since \( N \) satisfies exchange property by Theorem 2.18, there exist submodules \( C \subseteq A \) and \( D \subseteq B \) such that \( X = N \oplus C \oplus D \). Obviously, \( M \equiv X/N \equiv C \oplus D \). Thus \( E(M) \equiv E(C) \oplus E(D) \) where \( E(C) \) and \( E(D) \) are considered as submodules of \( E(A) \) and \( E(B) \), respectively. Note that there are injective submodules \( E_A \subseteq E(A) \) and \( E_B \subseteq E(B) \) for which \( E_A \oplus E(C) = E(A) \) and \( E_B \oplus E(D) = E(B) \). Now it is easy to see that \( E(N) \equiv E_A \oplus E_B \). By the hypothesis, we get \( \text{End}(E(X)) \equiv \text{End}(E(M)) \times \text{End}(E(N)) \), hence there exists a central idempotent \( f \in \text{End}(E(X)) \) for which \( f(E(X)) = E(M) \) and \( (1-f)(E(X)) = E(N) \). By Lemma 2.19, we obtain that \( f(E(A)) \equiv f(E(B)) \). As \( f(E(A)) = E(C) \) and \( f(E(B)) = E(D) \), a contradiction.

3. Classes of e-ADS modules and some ring conditions

Let \( \sigma[M] \) denote the Wisbauer category of a module \( M \), i.e. the full category of \( R\)-Mod consisting of submodules of quotients of direct sums of copies of \( M \) (see [14]).

**Theorem 3.1.** The following conditions are equivalent for a module \( M \):

1. \( M \) is semisimple.
2. Every module in \( \sigma[M] \) is e-ADS.
3. Every finitely generated module in \( \sigma[M] \) is e-ADS.
4. Every 4-generated module in \( \sigma[M] \) is e-ADS.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are clear.

(4) \( \Rightarrow \) (1) Let \( N \in \sigma[M] \) be a cyclic module and \( x \in M \). Then
\[
(N \oplus xR) \oplus (N \oplus xR)
\]
is a 4-generated module in \( \sigma[M] \) and hence is e-ADS by the hypothesis. By Lemma 2.8, \( N \oplus xR \) is automorphism \( N \oplus xR \)-invariant and \( N \) is \( xR \)-injective by Lemma 2.11. By [9, Theorem 1.4], \( N \) is \( M \)-injective. Thus \( M \) is semisimple by [4, Corollary 7.14].

Theorem 3.1 gives immediately the following.

**Corollary 3.2.** A ring \( R \) is semisimple Artinian if and only if every 4-generated \( R \)-module is e-ADS.

The following observation gives another characterization of e-ADS modules in the category \( \sigma[M] \).

**Theorem 3.3.** The following conditions are equivalent for a module \( M \):

1. \( M \) is semisimple.
2. The direct sum of every two e-ADS modules in \( \sigma[M] \) is e-ADS.
3. Every e-ADS module in \( \sigma[M] \) is \( M \)-injective.
4. The direct sum of any family of e-ADS modules in \( \sigma[M] \) is e-ADS.

**Proof.** (1) \( \Rightarrow \) (4) \( \Rightarrow \) (2). They are obvious.

(2) \( \Rightarrow \) (3) Let \( N \) be an e-ADS module. By our assumption, \( (N \oplus E_M(N)) \oplus (N \oplus E_M(N)) \) is e-ADS. Then \( N \oplus E_M(N) \) is automorphism invariant. Hence \( N \) is \( E_M(N) \)-injective by Lemma 2.11. It follows that \( N \) is \( M \)-injective.
(3) \(\Rightarrow\) (1) We consider a family \(\{S_i|i \in \mathbb{N}\} \subseteq \sigma[M]\) of simple right \(R\)-modules. It follows that \(\oplus_{i \in \mathbb{N}} S_i\) is semi-simple and so is e-ADS. By (3), \(\oplus_{i \in \mathbb{N}} S_i\) is \(M\)-injective. Therefore \(\oplus_{i \in \mathbb{N}} S_i\) is a direct summand of \(\oplus_{i \in \mathbb{N}} E_M(S_i)\). But \(\oplus_{i \in \mathbb{N}} S_i\) is essential in \(\oplus_{i \in \mathbb{N}} E_M(S_i)\) and then \(\oplus_{i \in \mathbb{N}} S_i = \oplus_{i \in \mathbb{N}} E_i\) is \(M\)-injective. Thus \(M\) is locally Noetherian. We can write \(E_M(M) = \oplus_{i \in \mathbb{N}} K_i\) for some indecomposable right \(R\)-modules \(K_i\) in \(\sigma[M]\) by [14, 27.4]. We have that every \(K_i\) is \(M\)-injective and obtain that every \(K_i\) is uniform. For each \(i \in I\), let \(0 \neq x \in K_i\). Since \(K_i\) is uniform, \(xR\) is uniform as well, hence \(xR\) is e-ADS. Then \(xR\) is \(M\)-injective by (3). It follows that \(xR\) is a direct summand of \(K_i\) and we have \(xR = K_i\). Thus \(K_i\) is simple for all \(i \in I\). That means \(E_M(M)\) is semi-simple. Thus \(M\) is semi-simple.

**Corollary 3.4.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is semi-simple Artinian.
2. The direct sum of every two e-ADS modules is e-ADS.
3. Every e-ADS module is injective.
4. The direct sum of any family of e-ADS modules is e-ADS.

We note that if \(M \oplus E(M)\) is e-ADS for an \(R\)-module \(M\), then \(M \cong E(M)\) by Theorem 2.9 and so \(M\) is injective.

**Theorem 3.5.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is right Noetherian.
2. The direct sum of injective right \(R\)-modules is e-ADS.
3. For any injective right \(R\)-module \(X\), \(X^{(N)}\) is e-ADS.

**Proof.** (1) \(\Rightarrow\) (2) They are obvious.

(3) \(\Rightarrow\) (1) Let \(X\) be an injective module. Clearly, \(X \oplus E(R_R)\) is also injective. Let \(M = X \oplus E(R_R)\). Since \(4 \cdot [N] = [N]\), we obtain that \((M^{(N)})^{(4)} \cong M^{(N)}\). By (3), \(M^{(N)} \oplus M^{(N)}\) is automorphism invariant. It follows that \(M^{(N)}\) is quasi-injective. On the other hand, \(X^{(N)}\) is isomorphic to a direct summand of \(M^{(N)}\). It implies that \(X^{(N)}\) is \(E(R_R)\)-injective and so \(X^{(N)}\) is injective. Hence \(R\) is right Noetherian. \(\square\)

A ring \(R\) is called a right V-ring if every simple right \(R\)-module is injective.

**Theorem 3.6.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is a right V-ring.
2. \(S \oplus E(S)\) is e-ADS for every simple right \(R\)-module \(S\).

**Proof.** (1) \(\Rightarrow\) (2) This is obvious.

(2) \(\Rightarrow\) (1) Assume that \(S \oplus E(S)\) is e-ADS for every simple right \(R\)-module \(S\). Let \(S\) be a simple right \(R\)-module. By the hypothesis, \(S \oplus E(S)\) is e-ADS. Then, by Theorem 2.9(1), \(S \cong E(S)\), and so \(S\) is injective. \(\square\)

**Theorem 3.7.** The following conditions are equivalent for a ring \(R\):

1. \(R\) is a QF-ring.
2. Every projective right \(R\)-module is e-ADS.
3. Every essential extension of any free right \(R\)-module is e-ADS.

**Proof.** (1) \(\Rightarrow\) (2) and (1) \(\Rightarrow\) (3) are obvious.

(2) \(\Rightarrow\) (1) Let \(I\) be a non-empty set. Clearly \((R^{(F)})^4\) is also a projective module. By (2), \((R^{(F)})^4 \oplus (R^{(F)})^4\) is automorphism invariant. It follows that \((R^{(F)})^4\) is quasi-injective. Therefore \((R^{(F)})^4\) is injective. Thus \(R\) is \(\Sigma\)-injective and so \(R\) is a QF-ring.
Let $F$ be a free right $R$-module. Then $F \oplus E(F)$ is an essential extension of a free right module $F^2$. By (3), $F \oplus E(F)$ is $\epsilon$-ADS, hence $F$ is injective. Now we have proved that every projective right $R$-module is injective. Thus $R$ is QF by the Faith-Walker theorem.

4. The Structure of $\epsilon$-ADS rings

We say that a ring $R$ is right $\epsilon$-ADS if it is an $\epsilon$-ADS module over itself. A right $\epsilon$-ADS ring $R$ is called trivial if $R_R$ is trivial $\epsilon$-ADS, i.e. the module $R_R$ does not have a decomposition $R_R = A \oplus B$ such that $E(A) \cong E(B)$. Otherwise $R$ is said to be a nontrivial $\epsilon$-ADS ring.

Let $R$ be a ring, $e$ be an idempotent of $R$, $S := eRe$ and $n \in \mathbb{N}$. Denote by $\mathcal{L}(eR^n)$ the lattice of all submodules of the projective $R$-module $eR^n$, and $\mathcal{L}(S^n)$ the lattice of all submodules of the free module $S^n$. Define two mappings

$$\Phi: \mathcal{L}(eR^n) \to \mathcal{L}(S^n)$$

and

$$\Psi: \mathcal{L}(S^n) \to \mathcal{L}(eR^n)$$

by the rules

$$\Phi(I) = I e; \quad \Psi(J) = JR$$

for arbitrary $I \in \mathcal{L}(eR^n)$ and $J \in \mathcal{L}(S^n)$.

**Lemma 4.1.** $\Phi$ and $\Psi$ are well-defined monotonic mappings. Moreover, $\Phi$ is a lattice homomorphism and $\Psi$ is compatible with the operation $+$.

**Proof.** Straightforward from the above notation. $\square$

Note that the inclusion $\Psi(J_1 \cap J_2) \subseteq \Psi(J_1) \cap \Psi(J_2)$ holds generally for arbitrary $J_1, J_2 \in \mathcal{L}(S^n)$ but the following example shows that the reverse need not be true.

**Example 4.2.** Let $R = \{(a_{ij}) \in M_{3 \times 3}(\mathbb{Q})| a_{31} = a_{32} = 0\}$ be a subring of matrix ring $M_{3 \times 3}(\mathbb{Q})$. Put $e := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $g := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $S := eRe$, $J_1 := fS$, and $J_2 := gS$. Then it is easy to see that

$$J_1 \cap J_2 = 0$$

and

$$J_1R \cap J_2R = \{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} | u, v \in \mathbb{Q} \}.$$ 

Thus $(J_1 \cap J_2)R \neq J_1R \cap J_2R$.

**Lemma 4.3.** Let $R$ be a ring and $e \in R$ be an idempotent such that $ReR = R$. Then $\Phi$ and $\Psi$ are mutually inverse lattice isomorphisms.

**Proof.** Let $S := eRe$. Since both $\Phi$ and $\Psi$ are monotonic, it is enough to show that $\Phi \Psi$ and $\Psi \Phi$ are identity mappings on $\mathcal{L}(S)$ and $\mathcal{L}(eR)$, respectively. Let $I \in \mathcal{L}(eR)$ and $J \in \mathcal{L}(S)$. Since $ReR = R$, we get

$$\Phi \Psi(I) = IeR = IReR = IR = I,$$

On the other hand $S = eRe$ and $J = Je$ imply that

$$\Phi \Psi(J) = JRe = JeRe = JS = J.$$
Recall that essentiality of modules can be expressed as a condition of lattices of submodules:

**Lemma 4.4.** Let \( A \subseteq B \) are submodules of a module \( M \). Then \( A \subseteq^e B \) if and only if there exists no submodule \( C \subseteq B \) such that \( A \cap C = 0 \).

**Proof.** This is well known. □

The following general consequence is a special case of [15, Theorem 1.2] for the lattice isomorphism from Lemma 4.3.

**Corollary 4.5.** Let \( R \) and \( S \) be rings, \( M \) an \( R \)-module, \( N \) an \( S \)-module and \( K, L \) submodules of \( M \). Suppose that \( \phi : \mathcal{L}(M_R) \to \mathcal{L}(N_S) \) is an isomorphism of lattices of all submodules of \( M \) and \( N \). Then \( K \) is a complement of \( L \) if and only if \( \phi(K) \) is a complement of \( \phi(L) \).

Lemmas 4.4, 2.1, 2.15 and Corollary 4.5 show that e-ADS, trivial e-ADS and relative automorphism invariant are lattice conditions. Thus the assertions of the following theorem hold true because lattices of all submodules of \( M \) and \( N \) are isomorphic.

**Theorem 4.6.** Let \( R \) and \( S \) be rings, \( M \) an \( R \)-module and \( N \) an \( S \)-module. Assume \( \phi : \mathcal{L}(M_R) \to \mathcal{L}(N_S) \) is an isomorphism of lattices.

1. \( M \) is (trivial) e-ADS if and only if \( N \) is a (trivial) e-ADS.
2. If \( M = A \oplus B \), then \( N = \phi(A) \oplus \phi(B) \) and \( A \) is B-automorphism invariant if and only if \( \phi(A) \) is \( \phi(B) \)-automorphism invariant.

Let \( n \in \mathbb{N} \) and \( e \) be an idempotent of a ring \( R \) such that \( ReR = R \). Recall that \( L(eR^n_R) \) and \( L(S^n_S) \) are isomorphic lattices by Lemma 4.3 for every \( n \in \mathbb{N} \), where \( S = eRe \).

**Theorem 4.7.** Let \( R \) be a ring, \( n \in \mathbb{N} \) and \( e \in R \) be an idempotent such that \( ReR = R \).

1. \( eR^n_R \) is a (trivial) e-ADS module if and only if \( eRe \) is (trivial) e-ADS as a right \( eRe \)-module.
2. \( eRe \) is B-automorphism invariant if and only if \( Ae \) is \( Be \)-automorphism invariant.
3. If \( S = eRe \), then \( S \) is automorphism invariant if and only if \( S^n_S \) is automorphism invariant.

**Proof.** (1) and (2) follow immediately from Theorem 4.6.

(3) It suffices to apply (2) for the decomposition \( eRe = eR \oplus eR \). □

The next observation shows that the class of e-ADS rings is closed under taking finite products.

**Proposition 4.8.** If \( R_1 \) and \( R_2 \) are e-ADS rings, then \( R_1 \times R_2 \) is e-ADS as well.

**Proof.** Put \( R := R_1 \times R_2 \) and let \( e_i \) be orthogonal central idempotents such that \( R_i = Re_i \) for \( i = 1, 2 \). It is easy to see that \( e_1 + e_2 = 1 \), \( E(R) = E(R_1) \oplus E(R_2) \) and \( E(R_i) = E(R)e_i \) for \( i = 1, 2 \). Suppose that \( R = A \oplus B \) is a module decomposition, \( C \subseteq A \), \( D \subseteq^e B \) and \( f : C \to D \) is an isomorphism. Then \( f_i : Ce_i \to De_i \) defined by \( f_i(r) = re_i \) is an isomorphism for each \( i = 1, 2 \). We note that for each \( Ce_i \subseteq^e A e_i \) and
$Dc_i \leq^e Be_i$ for each $i = 1, 2$. By the hypothesis, there exist extensions $g_i : A e_i \to Be_i$ of $f_i$. Clearly, $g = g_1 \oplus g_2 : A \to B$ extends $f$. □

We denote the set of all $n \times n$ matrices over a ring $B$ by $M_n(B)$.

**Lemma 4.9.** If $R$ is a non-trivial e-ADS ring, then there exists a right automorphism invariant ring $S$ such that $R \cong M_2(S)$.

**Proof.** Since $R$ is a non-trivial e-ADS ring, there exists an idempotent $e \in R$ for which $E(eR) \cong E((1 - e)R)$. Thus $eR \cong (1 - e)R$ is automorphism invariant by Theorem 2.9. Put $S := eRe$. Then

$$R \cong \text{End}(eR + eR) \cong M_2(S)$$

and $S$ is a right automorphism invariant ring by Theorem 4.7(3). □

Let $R$ be a ring. Recall that $R$ is said to be right non-singular if its right singular ideal $Z(R) = \{ r \in R : rI = 0 \text{ for some essential right ideal } I \text{ of } R \}$ is zero, and $R$ is called normal if if moreover its idempotents are central. Note that every abelian regular ring or every product of rings without non-trivial idempotents can serve as elementary examples of normal rings.

**Proposition 4.10.** Let $R$ be a right non-singular normal automorphism invariant ring. Then

(1) $R$ is trivial e-ADS,

(2) $M_2(R)$ is non-trivial e-ADS.

**Proof.** Denote by $Q$ the maximal right ring of quotients $R$. Obviously $eQ = E(eR$ for every idempotent $e$.

(1) As every central idempotent of $R$ is a central idempotent of $Q$, the assertion follows from Lemma 2.16.

(2) By Theorem 4.7 it is enough to prove that $M = R \oplus R$ is a non-trivial e-ADS module. Clearly, $M$ cannot be trivial. So it suffices to prove Theorem 2.10(4).

Suppose $R = e_1R \oplus f_1R$ for every $i = 1, 2$, where $(e_i, f_i)$ is a pair of orthogonal idempotents such that $e_1Q \oplus e_2Q \cong f_1Q \oplus f_2Q$. We claim that $A := e_1R \oplus e_2R \cong B := f_1R \oplus f_2R$ (and that $A$ is automorphism invariant).

Since $R$ is a normal ring, i.e., all idempotents $e_i, f_i$ of $R$, are central for each $i = 1, 2$, we have

$$e_iQ = e_i e_j Q \oplus e_i f_j Q$$

$$f_i Q = f_i e_j Q \oplus f_i f_j Q$$

for $i \neq j$. Hence $Q = e_1e_2Q \times e_1f_2Q \times f_1e_2Q \times f_1f_2Q$, where there is no nonzero homomorphism between two distinct components. Thus

$$E(A) = e_1Q + e_2Q \cong (e_1e_2Q)^{(2)} \oplus e_1f_2Q \oplus e_2f_1Q$$

and

$$E(B) = f_1Q + f_2Q \cong (f_1f_2Q)^{(2)} \oplus e_1f_2Q \oplus e_2f_1Q.$$ We have observed that $\text{Hom}(e_1e_2Q, E(B)) = 0$ as well as $\text{Hom}(e_1e_2Q, E(B)) = 0$ which implies that $e_1e_2 = 0 = f_1f_2$. Hence

$$E(A) \cong e_1f_2Q \oplus e_2f_1Q \cong E(B)$$

and so

$$A \cong e_1f_2R \oplus e_2f_1R \cong B.$$
Finally, since $e_1f_2R \oplus e_2f_1R$ is isomorphic to a direct summand of $R$ which is automorphism invariant, we obtain that $A$ is automorphism invariant by [8, Lemma 4].

We finish the section with the following criterion.

**Theorem 4.11.** Let $R$ be a right non-singular ring and $Q$ be its the maximal right ring of quotients. Then the following is equivalent:

1. $R$ is right e-ADS,
2. Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong M_2(S)$ for a suitable right automorphism invariant ring $S$,
3. Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong T \times M_2(S)$ for a suitable self-injective ring $T$ and a normal right automorphism invariant ring $S$.

**Proof.** (1) $\Rightarrow$ (2) If $R$ is a right trivial e-ADS ring, then $Q \cong E(R)$ has no a decomposition $Q = A \oplus B$ with a isomorphic summand, which implies that $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$.

If $R$ is a non-trivial e-ADS ring, then there exists a right automorphism invariant ring $S$ such that $R \cong M_2(S)$ by Lemma 4.9.

(2) $\Rightarrow$ (3) Assume $R \cong M_2(S_0)$ for a right automorphism invariant ring $S_0$. Clearly, $S_0$ is, moreover, non-singular, hence there exists a right self-injective ring $S_1$ and a normal right automorphism invariant ring $S$ such that $S_0 \cong S_1 \times S$ by [5, Theorem 7]. Now it is easy to see that $M_2(S_0) \cong M_2(S_1) \times M_2(S)$ and $T = M_2(S_0)$ is self-injective by [7, Corollary 9.3].

(3) $\Rightarrow$ (1) We remark that the first condition implies that $R$ is a trivial e-ADS ring. Suppose that $R \cong T \times M_2(S)$ where $T$ is a self-injective ring and $S$ is a normal right automorphism invariant ring. Note that $T$ is an e-ADS ring and $M_2(S)$ is e-ADS by Lemma 4.10. So, $R$ is right e-ADS by Lemma 4.8.

**Corollary 4.12.** Every simple non-trivial right e-ADS ring is necessarily self-injective.

**Proof.** It follows from Theorem 4.11 and [5, Corollary 10].

**References**

5. N. Er, S. Singh, A. K. Srivastava: Rings and modules which are stable under automorphisms of their injective hulls, J. Algebra, 379 (2013), 222-229.