ADS ABELIAN GROUPS

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Abstract. The subgroup $G \leq A$ is ADS if, for every G-high $H \leq A$ (i.e., maximal with respect to the property $G \cap H = 0$), we have $A = G \oplus H$, and A itself is an ADS group if all of its summands inherit this property. In the present paper, we characterize the structure of ADS abelian groups, which can be defined as groups such that members of the decomposition into two direct summands are mutually injective. In particular, an ADS abelian group is either divisible, or a direct sum of an indecomposable torsion-free group and a divisible torsion group, or a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of Prüfer groups.

Introduction

Various decomposition properties play key role in structural module theory. A classical example presents a notion of a projective module which can be described either as a direct summand of a free module or as a direct sum of countable generated projective modules. One of the useful decomposition properties relating (and generalizing) the concept injectivity is called the absolute direct summand (ADS) condition. Recall that a module M is said to be *ADS* whenever S is a summand of A and T is maximal with respect to $S \cap T = 0$, then T is a complement of S, i.e, $A = S \oplus T$. The notion of ADS abelian group was introduced by Laszlo Fuchs (see e.g. [5]) and its importance has been shown in many later works (e.g. [1, 2, 8, 10]), which, among other things, describe its structural properties over particular classes of rings.

This note deals with the structural description of an ADS abelian group. While description of a torsion-free ADS module over general domain as either an indecomposable or injective module is easy (Theorem 1.3), The characterization of torsion ADS abelian group applies [1, Proposition 3.2] and the case of mixed ADS abelian groups, in addition, needs classical Kulikov's result on decomposability of mixed abelian groups [9, Theorem 7]. Finally, we prove a structural description of ADS abelian groups (Theorem 3.1) which assert that a non-divisible ADS abelian group is either a direct sum of an indecomposable torsion-free group and a divisible torsion group, or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of Prüfer groups $\mathbb{Z}_{p^\infty}$.

Throughout this paper, R is an associative ring with unity and all modules over R are unitary right modules. We also write M_R to indicate that M is a right R-module. For a submodule N of M, we use $N \leq M$ to mean that N is a submodule of M. We write $\mathbb{Z}$ and $\mathbb{N}$ for the ring of integers and for the set of all positive integers numbers, respectively. For any group G, as usually $X \subseteq G$ shows X is a

Date: April 14, 2023.
2000 *Mathematics Subject Classification.* 20K40 (20K20, 20K21).
Key words and phrases. Abelian groups, C2 modules, absolute direct summand (ADS) .

subset of G but $X \leq G$ is used only for a subgroup X of G. For unexplained notions and results, we refer the reader to [9, 5, 6, 7].

1. Torsion-free ADS modules

We start by two useful observations, which list some important properties of ADS modules and mutually injective modules.

Proposition 1.1. *Let M be an R-module.*

1. *M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.*
2. *The ADS property is inherited by direct summands*
3. *If M is indecomposable or injective, then it is ADS.*

Proof. (1) This is [1, Proposition 3.2].
(2) Assume M is ADS and $M = N \oplus K$ where $N = A \oplus B$ for some modules N, K, A and B. Then $M = A \oplus B \oplus K$. Since M is ADS, the modules A and B are mutually injective by (1) which implies N is ADS.
(3) An indecomposable module M is ADS since 0 and M are mutually injective by (1), and an injective module M is ADS by (2) because the injective property is inherited by direct summands. □

Proposition 1.2. *Let A be a B-injective module over a ring R and C be a submodule of B.*

1. *A is C-injective.*
2. *If $C \cong R$, then A is injective.*

Proof. (1) Let D be a submodule of C and $f \in \mathrm{Hom}(C, A)$. Since f can be extended to a homomorphism $\tilde{f} \in \mathrm{Hom}(B, A)$, the restriction of $\tilde{f}|_C \in \mathrm{Hom}(C, A)$ is a desired extension of f.
(2) It is an easy consequence of the Baer's criterion of the injectivity. □

An R-module M is said to be torsion-free if, for all $r \in R$ and $m \in M$, the annihilation $mr = 0$ is possible only if m is a linear combination $\sum m_i r_i$ of certain $m_i \in M$ with coefficients $r_i \in R$ already annihilated by r, i.e. such that $r_i r = 0$.

As usual, let us denote by

$$\tau(M) = \{m \in M \mid \exists r \in R \setminus \{0\} \text{ such that } mr = 0\}$$

the torsion part of M.

Theorem 1.3. *Let M be a torsion-free module over a domain R. Then M is ADS if and only f it is either indecomposable or injective.*

Proof. Let M be a decomposable ADS torsion free module. Then there exists a non-trivial decomposition $M = A \oplus B$. Since arbitrary non-zero cyclic submodule of A and B is free, both groups A and B and so M are injective by Proposition 1.2.

The reverse implication follows from Proposition 1.1. □

Recall that injective abelian groups are precisely divisible groups and formulate a straightforward consequence of the previous assertion.

Corollary 1.4. *Let G be a torsion-free group. Then G is ADS if and only if it is either indecomposable or divisible.*

It is well known that $\tau(M)$ forms a submodule of a module M over an arbitrary domain R.

Theorem 1.5. *Let R be a domain. The followings are equivalent for a non-zero torsion-free R-module M and a torsion R-module T:*

(1) *$M \oplus T$ is ADS;*
(2) *T is injective and M is either indecomposable or injective.*

Proof. (1) ⇒ (2) Let $M \oplus T$ be an ADS module. By Propositions 1.1(1) and 1.2, the module T is injective. Since M is torsion-free ADS, it is either indecomposable or injective by Theorem 1.3.
(2) ⇒ (1) If both M and T are injective, then $M \oplus T$ is ADS by Proposition 1.1.

Let M be an indecomposable torsion-free module and T an injective torsion module. Assume $M \oplus T = A \oplus B$. The observations $\tau(A) = A \cap T$ and $\tau(B) = B \cap T$ yield $T = \tau(A) \oplus \tau(B)$. Hence $\tau(A)$ and $\tau(B)$ are injective submodules of A and B, respectively. Hence they are direct summands which means that there exist submodules C and D for which $A = \tau(A) \oplus C$ and $B = \tau(B) \oplus D$. As

$$M \cong (M \oplus T)/(\tau(A) \oplus \tau(B)) \cong (A/\tau(A)) \oplus (B/\tau(B) \cong C \oplus D$$

is an indecomposable module, one of the summands is zero. Without loss of generality suppose that $D = 0$. Hence $C \cong M$. Since $\tau(A)$ and $B = \tau(B)$ are injective, we get $A \cong M \oplus \tau(A)$, and $\mathrm{Hom}(N, M) = 0$ for each subgroup N of B. Now it is easy to see that A and B are mutually injective, which finishes the proof. □

The following is a consequence of Theorem 1.5 and well known facts

(i) every finitely generated abelian group is isomorphic to a direct sum of a free and a torsion group;

(ii) every finitely generated torsion group is finite.

Corollary 1.6. *An infinite cyclic group is the only example of a finitely generated infinite abelian group which is ADS.*

Proof. If A is an infinite finitely generated ADS group. then $A \cong F \oplus T$ for a non-zero free group F and a finite torsion group T. Note that every non-zero injective abelian group is infinitely generated, hence $T = 0$ and F is indecomposable by Theorem 1.5. Thus $M \cong F \cong \mathbb{Z}$. □

2. Mixed ADS abelian groups

Let $\mathbb{P}$ denote the set of all prime numbers, $p \in \mathbb{P}$ and $a \in A$. Then a is said to be p-torsion provided there exists a natural number n such that $p^n a = 0$.

Denote by

$l_p(a) = \min\{n \mid p^n a = 0\}$ the p-length of an element a,

$\tau_p(A)$ the subgroup of all p-torsion elements,

$l_p(A) = \sup\{l_p(a) \mid a \in \tau_p(A)\} \in \mathbb{N} \cup \{\infty\}$.

It is well known that $\tau(A) = \bigoplus_{p \in \mathbb{P}} \tau_p(A)$.

Lemma 2.1. *Groups $\mathbb{Z}_{p^n}^{(\kappa)}$ and $\mathbb{Z}_{p^n}^{(\lambda)}$ are mutually injective for any $n \in \mathbb{N}$, and non-zero cardinals κ and λ.*

Proof. Put $A = \mathbb{Z}_{p^n}^{(\kappa)}$ and $B = \mathbb{Z}_{p^n}^{(\lambda)}$. It suffices to prove that B is A-injective. Suppose that C is a subgroup of A and $f \in \operatorname{Hom}(C, B)$. Then f can be extended to a homomorphism $\tilde{f} : A \to E(B) = Z_{p^\infty}^{(\lambda)}$. Since $l_p(\tilde{f}(A)) \leq l_p(A) = n$, we get $\tilde{f}(A) \subseteq B$, i.e. B is A-injective. □

Lemma 2.2. *Let A and B be mutually injective abelian groups, $n := l_p(A)$ and $m := l_p(B) \in \mathbb{N} \cup \{\infty\}$. Then $n = m$ and there exist cardinals κ and λ such that $\tau_p(A) \cong Z_{p^n}^{(\kappa)}$ and $\tau_p(B) \cong Z_{p^n}^{(\lambda)}$.*

In particular, if $n = \infty$, then $\tau_p(A)$ and $\tau_p(B)$ are p-divisible direct summand of A and B.

Proof. Without loss of generality, we may suppose that $A = \tau_p(A)$ and $B = \tau_p(B)$.
Case 1. Assume m is finite. First, we show that every element of A is contained in a cyclic submodule of p-length at least m. Fix $a \in A$. If $l_p(a) \geq m$ there is nothing to prove, so suppose $l_p(a) < m$. Since there exists $b \in B$ of p-length m, there exists an element $c \in \langle b \rangle$ such that $l_p(b) = l_p(a)$. Hence $\langle c \rangle$ and $\langle a \rangle$ are isomorphic and so this isomorphism can be extended to an injective homomorphisms $f : \langle b \rangle \to A$ by Lemma1.2. As $l_p(b) = m$ and $a \in \langle b \rangle$, we are done.
Case 2. Assume n is finite. Clearly, the symmetric argument proves that $n = m$, and so every element of A and B is contained in a cyclic submodule of p-length equal to n. Since groups A and B are bounded, they are direct sums of cyclic groups by [7, Theorem 3.5.2]. Thus there exists cardinals κ and λ for which $A \cong \mathbb{Z}_{p^n}^{(\kappa)}$ and $B \cong \mathbb{Z}_{p^n}^{(\lambda)}$.
Case 3. Assume $m = \infty$. By the same argument as in the finite case, we can obtain that every element of A is contained in a cyclic submodule of an arbitrary large p-length, so A and, by the symmetric argument, B are p-divisible. Hence there exists cardinals κ and λ for which $A \cong \mathbb{Z}_{p^\infty}^{(\kappa)}$ and $B \cong \mathbb{Z}_{p^\infty}^{(\lambda)}$. □

An abelian group A is said to be proper mixed provided $0 \neq \tau(A) \neq A$.

Proposition 2.3. *The followings are equivalent for a non-divisible proper mixed abelian group G:*

(1) *G is ADS;*
(2) *$G \cong A \oplus B$, where A is non-divisible indecomposable torsion-free and B is divisible torsion.*

Proof. (1) ⇒ (2) Let G be ADS. As any proper mixed abelian group is decomposable by [9, Theorem 7], we can fix a nontrivial decomposition $G = A \oplus B$.
Case 1. Assume that both A and B are not torsion. Then A, B and so G is divisible by Lemma 1.2, which contradicts to the hypothesis.
Case 2. Assume that A is non-torsion and B torsion. Applying Proposition 1.2(2) we obtain that B is a divisible torsion group. Since A and B are mutually injective, we get $\tau_p(A)$ is divisible for each $p \in \mathbb{P}$ satisfying $\tau_p(B) \neq 0$ by Lemma 2.2. Hence, for each such p, the components $\tau_p(M) = \tau_p(A) \oplus \tau_p(B)$ are divisible. Now we may suppose without loss of generality that B contains all p-torsion components of G

which are divisible.
Case 3. Now, assume that A is proper mixed. Then Theorem 7 in [9] ensures a non-trivial decomposition $A = C \oplus D$, where C is non-torsion and D is torsion. We have shown that D is necessarily divisible, hence it is contained in B, a contradiction with the non-triviality of the decomposition. Thus A is a non-divisible torsion-free ADS group, so it is indecomposable by Theorem 1.3.
(2) $\Rightarrow$ (1) This implication follows from Theorem 1.5. □

Since there is no homomorphism between any two different p-components of a torsion ADS abelian group, we need to describe single p-components.

Proposition 2.4. *The followings are equivalent for a torsion group G:*

1. *G is ADS;*
2. *For every $p \in \mathbb{P}$, there exists $n \in \mathbb{N} \cup \{0, \infty\}$ and a cardinal κ for which $\tau_p(M) \cong \mathbb{Z}_{p^n}^{(\kappa)}$.*

Proof. (1) $\Rightarrow$ (2) As $G = \bigoplus_{p \in \mathbb{P}} \tau_p(M)$ and $\mathrm{Hom}(N, \tau_q(M)) = 0$ for each pair of distinct primes p, q and every subgroup H of $\tau_p(G)$, it is enough to prove the assertion for one p-component. Thus suppose without loss of generality that G is a p-group.

Let G be an ADS p-group.
Case 1. If G is divisible, then $G \cong \mathbb{Z}_{p^\infty}^{(\kappa)}$ and we are done.
Case 2. Assume G is not divisible. By [9, Theorem 5], G contains pure cyclic subgroup C which is a direct summand of G. If $C = G$ there is nothing to prove, otherwise direct complement of C is isomorphic to a direct product of copies of C be Lemma 2.2. Hence there exists a cardinal κ such that $G \cong \mathbb{Z}_{p^n}^{(\kappa)}$ for $n = l_p(C)$.
(2) $\Rightarrow$ (1) Let $G \cong \mathbb{Z}_{p^n}^{(\kappa)}$ for some n and κ and consider a decomposition $G = A \oplus B$. Then $A \cong \mathbb{Z}_{p^n}^{(\lambda_A)}$ and $B \cong \mathbb{Z}_{p^n}^{(\lambda_B)}$ for suitable cardinals λ_A, λ_B. Hence A and B are mutually injective by Lemma 2.1. □

As an easy application of the previous assertion and Chinese reminder theorem we get the characterization of finite ADS abelian groups.

Corollary 2.5. *Let F be a finite abelian group and $n\mathbb{Z} = \mathrm{Ann}(F)$. Then F is ADS if and only if F is a projective $\mathbb{Z}/n\mathbb{Z}$-module.*

3. The criterion and examples

Now we are ready to formulate our main result.

Theorem 3.1. *An abelian group is ADS if and only if*

1. *either it is divisible,*
2. *or it is a direct sum of an indecomposable torsion-free group and a divisible torsion group,*
3. *or it is a torsion group such that each p-component is a direct sum of cyclic p-groups of the same length or of $\mathbb{Z}_{p^\infty}$.*

Proof. If a group G is torsion-free ADS, then (1) or (2) holds with the trivial torsion part by Theorem 1.3. If G is proper mixed ADS, then holds (2) holds by Proposition 2.3. If G is torsion ADS, then (3) holds by Proposition 2.4.

The reverse implication follows from Theorem 1.3 and Propositions 2.3 and 2.4. □

Recall that there exist arbitrary large examples of non-injective torsion-free abelian group which satisfies the ADS condition.

Example 3.2. For an arbitrary cardinal κ, by [4, Corollary 1], there exists 2^κ indecomposable torsion-free abelian groups of cardinality κ, all of which will reduced ADS groups.

Finally, compare the class of ADS and C2 groups.

The (abelian) group G is C2 if whenever A is a summand of G and B is a subgroup of G isomorphic to A, then B is also a summand of G [3]. By [3];
(i) every divisible group is injective (so quasi-injective) so C2,
(ii) a torsion-free group is C2 iff it is divisible,
(iii) the only indecomposable C2 groups are the cocyclic groups and $\mathbb{Q}$,
(iv) a torsion group is C2 iff it has homococyclic (if it is the direct sum of copies of a single cocyclic group (i.e., either $\mathbb{Z}_{p^\infty}$ or $\mathbb{Z}_{p^k}$ for some positive integer k)) or divisible p-components.

Example 3.3. Since the infinite cyclic group is ADS by Corollary 1.6 and finite cyclic groups have cyclic p-components so they are ADS by Propositions 2.3, each cyclic group is ADS. Note that the same result follows directly from the observation that every subgroup of a cyclic group is fully invariant. Furthermore, infinite cyclic group is not C2.

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