Variants of absolute direct summand property

J. Žemlička

12th International Algebraic Conference in Ukraine,
July 2–6, 2019, Vinnytsia
ADS-modules

Type-ADS modules

Essentially ADS modules

Rings
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A right module $M$ over $R$ is called ADS (absolute direct summand) if $M = S \oplus T'$ for every submodules $S$, $T$, $T'$ such that $M = S \oplus T$ and $T'$ is a complement of $S$. 

Example (1) If every idempotent of $R$ is central (in particular if $R$ is commutative or reduced), then $R$ is ADS.

Example (2) Every cyclic module over commutative ring is ADS.
- In the sequel $R$ denotes an associative ring with unit and $M$ a right $R$-module.

- A right module $M$ over $R$ is called ADS (absolute direct summand) if $M = S \oplus T'$ for every submodules $S$, $T$, $T'$ such that $M = S \oplus T$ and $T'$ is a complement of $S$.

**Example**

1. If every idempotent of $R$ is central (in particular if $R$ is commutative or reduced), then $R_R$ is ADS.
2. Every cyclic module over commutative ring is ADS.
A module $A$ is \textit{B-injective} if every homomorphism $C \to A$ for every submodule $C \leq B$ can be extended to a homomorphism $B \to A$.

\textbf{Theorem (Alahmadi, Jain Leroy, 2012)}

\textit{The following is equivalent:}

1. $M$ is ADS
2. $A$ and $B$ are mutually injective modules for every $M = A \oplus B$,
3. $A$ is a $bR$-injective module for every $M = A \oplus B$ and $b \in B$. 

\textbf{Theorem (Alahmadi, Jain Leroy, 2012)}

Let $R$ be an simple ring. If $R$ is ADS, then either $R$ is indecomposable or $R$ is a right self-injective regular ring.
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- $M = A \oplus B$ is a *type decomposition*, if $A$ and $B$ are type submodules of $M$. 
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An $R$-module $M$ is type-ADS if for every type decomposition $M = A \oplus B$ and every arbitrary type complement $C$ of $A$, we have $M = A \oplus C$.
Example

(1) Every ADS module is type-ADS.

(2) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field. Then $R_R$ is type-ADS, however it is not ADS.
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Corollary A type direct summand of a type-ADS module is type-ADS.
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Corollary
A type direct summand of a type-ADS module is type-ADS.
Theorem (Abdioğlu, Ž. 2018)

The following is equivalent:

(1) $M$ is type-ADS.

(2) $\alpha(M) \leq M$ for all idempotents $\alpha \in \text{End}(E(M))$ such that $(1-\alpha)(E(M)) \cap M$ is a type direct summand of $M$.

(3) For every decomposition $E(M) = E_1 \oplus E_2$ where $E_1 \cap M$ is a type direct summand of $M$, $M = (E_1 \cap M) \oplus (E_2 \cap M)$.

A submodule $X$ of $M$ is called fully invariant if for every $f \in \text{End}(M)$, $f(X) \leq X$. 

Lemma

Let $M = \bigoplus_{i \leq n} M_i$. If each $M_i$ is type-ADS fully invariant submodule of $M$ and $M_i$ is $\bigoplus_{j \neq i} M_j$-injective for all $i$, then $M$ is type-ADS.

Example

Let $M_1 = \mathbb{Z}$ and $M_2 = \mathbb{Z}^2$ be $\mathbb{Z}$-modules. Then $M_1$ and $M_2$ are indecomposable, hence type-ADS, but $M = M_1 \oplus M_2$ is not type-ADS.
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$M$ is called an essentially ADS-module if $M = S \oplus T'$ for each decomposition $M = S \oplus T$ and each complement $T'$ of $S$ with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq^e S$. 

Theorem (Koşan, Quynh, Ž. 2019)

Let $M$ be an $R$-module.

1. If $E(A) \cong E(B)$ for each decomposition $M = A \oplus B$, then $M$ is $e$-ADS.

2. If $M$ is an $e$-ADS module with a decomposition $M = A \oplus B$ such that $E(A) \cong E(B)$, then $A \cong B$ and the modules $A$ and $B$ are automorphism invariant.

Example

1. Every ADS module is $e$-ADS.

2. Let $T$ be a non-divisible torsion abelian group and $M = \mathbb{Z} \oplus T$. Since $E(A) \cong E(B)$ for every $M = A \oplus B$, $M$ is an $e$-ADS abelian group and it is not ADS, since $T$ is not $\mathbb{Z}$-injective.

3. Let $M = \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p^2$ for some prime $p$. Then $M$ is $e$-ADS and $\mathbb{Z}_p \oplus \mathbb{Z}_p^2$ is not $e$-ADS.
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**Theorem (Koşan, Quynh, Ž. 2019)**

Let $M$ be an $R$-module.

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Theorem (Koşan, Quynh, Ž. 2019)

*The following is equivalent:*

1. \( M \) is e-ADS.
2. For every decomposition \( M = S \oplus T \), if \( T' \) is a complement of \( S \) in \( M \) and \( T \) is a complement of \( T' \) in \( M \), then \( M = S \oplus T' \).
3. \( A \) and \( B \) are relatively automorphism invariant for each decomposition \( M = A \oplus B \).

**Lemma**

Let \( M \) be an e-ADS module. If \( M \) has a decomposition \( M = A \oplus B \) such that \( E(A) \cong E(B) \), then \( A \) is e-ADS.

A module \( M \) is trivial e-ADS if it has no decomposition \( M = A \oplus B \) such that \( E(A) \cong E(B) \).

**Lemma**

\( M \) is trivial e-ADS if and only if for every decomposition \( M = A \oplus B \) no complement of \( A \) is a complement of \( B \).
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$M$ is trivial e-ADS if and only if for every decomposition $M = A \oplus B$ no complement of $A$ is a complement of $B$. 
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1. $R$ is right e-ADS,
2. Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong M_2(S)$ for a suitable right automorphism invariant ring $S$, 
3. Either $eQ \not\cong (1-e)Q$ for any idempotent $e \in R$ or $R \cong T \times M_2(S)$ for a suitable self-injective ring $T$ and a normal right automorphism invariant ring $S*.}
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