$w = y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6)$ is a smooth WEP, $E(K) = V_w(K) \cup \{\infty\}$ is equipped by the operations $\oplus, \ominus$.

<u>Theorem 8.8</u> Let w be smooth at $V_w(K)$. Then $(E(K), \oplus, \ominus, \infty)$ is a commutative group. If $\gamma = (\gamma_1, \gamma_2), \delta = (\delta_1, \delta_2), \eta = (\eta_1, \eta_2) \in V_w(K)$, then (1) $\ominus \gamma = (\gamma_1, -\gamma_2 - a_1\gamma_1 - a_3)$

(2) if $\gamma \neq \ominus \delta$ and $\eta = \gamma \oplus \delta \Rightarrow$

$(\eta_1, \eta_2) = (-\gamma_1 - \delta_1 + \lambda^2 + a_1\lambda - a_2, \lambda(\gamma_1 - \eta_1) - \gamma_2 - a_1\eta_1 - a_3)$ where $\lambda = \frac{\delta_2 - \gamma_2}{\delta_1 - \gamma_1}$ if $\gamma_1 \neq \delta_1$ or $\lambda = \dfrac{3\gamma_1^2 + 2a_2\gamma_1 - a_1\gamma_2 + a_4}{2\gamma_2 + a_1\gamma_1 + a_3}$ if $\gamma_1 = \delta_1$

<u>proof</u>: by the definition $E(K) \to \mathbb{P}^{(1)}_{L/K}$ is a bijection compatible
8.6(3) $\quad \gamma \to P_\gamma$ with $\oplus$ & $\ominus$

$\Longrightarrow E(K) \cong \mathrm{Pic}^0(L/K)$ is a commutative group.

Note that (*) $\gamma + \delta = \eta \overset{\text{Observ}(6)}{\Longleftrightarrow} [P_\gamma + P_\delta] = [P_\eta + P_\infty]$

(1) Let $\hat{\ell} := x - \gamma_1 \in K[x,y] \overset{8.7(1)}{\Longrightarrow} \exists!\, \delta = (\gamma_1, \delta_2) : [P_\gamma + P_\delta] = [2P_\infty]$

$\overset{(*)}{\Longrightarrow} \gamma + \delta = \infty$ i.e. $\delta = \ominus\gamma$; $\delta_2 = (\gamma_1, -\gamma_2 - a_1\gamma_1 - a_3)$ by 8.7(1)

(2) Let $\gamma, \delta \in V_w(K)$, $\gamma \neq \ominus\delta$, we define line $\hat{\ell}$:

(a) $\hat{\ell} := t_\gamma(w)$ if $\gamma = \delta$ (b) $\hat{\ell} = y - \frac{\gamma_2 - \delta_2}{\gamma_1 - \delta_1} x + \frac{\gamma_1\delta_2 - \delta_1\gamma_2}{\gamma_1 - \delta_1}$ if $\gamma \neq \delta$

$\overset{8.7(2)}{\Longrightarrow} \exists\, \tilde{\eta} \in V_w(K) : [P_\gamma + P_\delta + P_{\tilde{\eta}}] = [3P_\infty] \Longrightarrow \gamma + \delta + \tilde{\eta} = \infty$

Note $\frac{\partial w}{\partial y}(\gamma) = 2\gamma_2 + a_1\gamma_1 + a_3 = 0 \overset{8.7(1)}{\Longleftrightarrow} \delta = \ominus\gamma \Rightarrow \gamma_1 = \delta_1 \leftrightarrow \delta \in \{\gamma, \ominus\gamma\}$

Then $\hat{\ell} : y - \lambda x - \mu$ where (a) $\boxed{\lambda = \frac{\partial w}{\partial x}(\gamma) \Big/ \frac{\partial w}{\partial y}(\gamma)}$ for $\gamma = \delta$

(b) $\boxed{\lambda = \frac{\delta_2 - \gamma_2}{\delta_1 - \gamma_1}}$ for $\gamma \neq \delta$

$\overset{8.7(7)}{\Longrightarrow} \tilde{\eta} = (\underbrace{-\gamma_1 - \delta_1 + \lambda^2 + a_1\lambda - a_2}_{\tilde{\eta}_1}, \lambda\tilde{\eta}_1 + \mu)$, then put $\eta = \ominus\tilde{\eta}$

$\Rightarrow \lambda x_1 + \mu = -(\lambda x_1 + \mu) + a_1 x_1 - a_3$ as $\hat{\ell}(\gamma_1, \gamma_2) = 0$

$\Rightarrow \eta = (-\gamma_1 - \delta_1 + \lambda^2 + a_1\lambda - a_2,\ \lambda(\gamma_1 - \eta_1) - \gamma_2 + a_1\eta_1 - a_3)$ $\mathfrak{z}(\eta)$

<u>Corollary 8.9</u> If $K \subseteq F \subseteq \bar{K} \Rightarrow E(K)$ is a subgroup of $E(F)$

<u>Example 8.10</u> Let $y^2 = x^3 + 1 \in \mathbb{F}_5[x, y]$ be WE

it is ~~s~~ smooth by 3.12.

$E(\mathbb{F}_5) = \{(0,1), (0,4), (4,0), (2,2), (2,3), \infty\}$ $(\cong \mathbb{Z}_6)$

$(0,1) \oplus (0,4) = \infty = (4,0) \oplus (4,0) = \infty = (2,2) \oplus (2,3)$

Compute $\underbrace{(0,4) \oplus (4,0)}$ $= (\underbrace{0 - 4 + (-1)^2}_{\eta_2}, 4(0-2) - 4) = \underline{(2,3)}$

$a_1 = a_3 = a_2 = a_4 = 0$ $\quad \lambda = \frac{0-4}{4-0} = 4 (= -1)$

$(a_6 = 1)$

9. Projective curves

Let $n \geq 1$, K be a field, $\bar{K}$ an algebraic closure of K

T&N Denote $a = (a_0 : a_1 : \dots : a_n) = \mathrm{Span}_K((a_0, a_1, \dots, a_n)) \subseteq K^{n+1}$,
then a is a <u>projective point</u> with homogeneous cordinates $(a_0 : \dots$

$\mathbb{P}^n(K) := \{(a_0 : a_1 : \dots : a_n) \mid (a_0, \dots a_n) \in K^{n+1} \setminus \{\underline{0}\}\}$, $\mathbb{P}^n := \mathbb{P}^n(\bar{K})$

— " — is called a <u>projective space</u> of dimension n

$F \in K[x_0, x_1, \dots x_n]$ is called a <u>homogeneous polynomial</u> of degree $d \geq 0$ if $F \in H_d := \mathrm{Span}_K\{x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} \mid \sum_{j=0}^{n} i_j = d\}$

(i.e. $\deg F = \mathrm{mult}\, F = d$ or $F = 0$)

$K[X_0, X_1, \dots, X_n] = \bigcup_{d \geq 0} H_d \; (\subseteq K[x_0, x_1, \dots x_n])$ denotes the set of all homogeneous polynomials

$\boxed{\text{T\&N}}$ $K(\mathbb{P}^n) := \{0\} \cup \{\frac{F}{G} \mid \exists d \geq 0 : F, G \in H_d \subseteq k[X_0, \dots, X_n]\} \subseteq K(x_0, \dots, x_n)$

Let $F \in k[X_0, \dots, X_n]$; $F(a) := F(a_0, \dots, a_n)$ for $a = (a_0 : \dots : a_n)$

F is <u>smooth</u> at $a \in \mathbb{P}^n$ if $\exists j : \frac{\partial F}{\partial X_j}(a) \neq 0$,

<u>singular</u> at $a \in \mathbb{P}^n$ otherwise.

a is a <u>homogeneous zero</u> of F : $F(a) = 0$

Let $M \subseteq k[X_0, \dots, X_n]$, then $V_M = \{a \in \mathbb{P}^n \mid F(a) = 0 \ \forall F \in M\}$

$V_F := V_{\{F\}}$, $V_M(k) := V_M \cap \mathbb{P}_n(k)$ (= k-rational projective points)

if F is irreducible, then $\boxed{K(V_F) := \{\frac{G + (F)}{H + (F)} \mid G, H \in k[X_0 \dots X_n], \deg G = \deg H\}}$

V_M is a <u>projective affine set</u>, V_F - <u>projective irreducible curve</u> if $F \in k[X_0, X_1, X_2]$

(i.e. $V_F \subseteq \mathbb{P}^2$)

<u>Observation A</u> Let $d \geq 0$, $i_0, \dots, i_n \geq 0$, $F \in H_d$ ($\subseteq k[X_0, \dots X_n]$)

(1) if $\sum_{j=0}^{n} i_j = d \Rightarrow \sum_{r=0}^{n} \frac{\partial \prod_{j}^{n} X_j^{i_j}}{\partial X_r} = d \prod_{j=0}^{n} X_j^{i_j} \Rightarrow$ (2) $\sum_{r=0}^{n} \frac{\partial F}{\partial X_r} = d F$.

(3) $K(\mathbb{P}^n)$ is a subfield of $K(x_0,\dots,x_n)$ containing K

(4) $K(V_F)$ ——— " ——— the fraction field of $K[x_0\dots x_n]/(F)$.

$\boxed{\text{Def N}}$ Let $f \in K[x_1 \dots x_n] - \{0\}$: $\hat{f} := X_0^{\deg f} f\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \dots \frac{X_n}{X_0}\right)$

$\hat{0} := 0$

$\forall j \geq 0$ define $\pi_j : K[X_0, \dots, X_n] \to K[x_0, \dots, x_{j-1}, x_{j+1}, \dots x_n]$

$\pi_j(F) := F(x_0, x_1, \dots, x_{j-1}, 1, x_{j+1}, \dots x_n)$

$\forall a = (a_1, \dots, a_n) \in \mathbb{A}^n$ define $\hat{a} := (1 : a_1 : a_2 : \dots : a_n) \in \mathbb{P}^n$

$\forall j \geq 0$ denote $p_j : \mathbb{P}^n \dashrightarrow \mathbb{A}^n$ a partial mapping defined by

$p_j((a_0 : a_1 : \dots a_n)) := \left(\frac{a_0}{a_j}, \dots \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \dots \frac{a_n}{a_j}\right)$ if $a_j \neq 0$

<u>Observation B</u> Let $f, g \in K[x_1, \dots x_n]$;

(1) $\hat{f} \in K[X_0, \dots X_n]$, $\widehat{fg} = \hat{f}\hat{g}$, $\pi_0(\hat{f}) = f$,

(2) if $0 \notin \{fg, f+g\} \Rightarrow X_0^{\deg g}\hat{f} + X_0^{\deg f}\hat{g} = X_0^{s}\widehat{(f+g)}$ for $s = \deg f + \deg g - \deg(f+g)$

(3) f is irreducible $\Leftrightarrow$ $\hat{f}$ is irreducible,

(4) $a \in V_f \Leftrightarrow \hat{a} \in V_{\hat{f}}$ for $a \in \mathbb{A}^n$,

(5) $p_0(V_{X_0^\lambda \hat{f}}) = V_f \quad \forall \lambda \geq 0$.

<u>Lemma 9.1</u> If $f \in K[x_1, \ldots, x_n]$ and $a \in V_f$. Then

f is smooth at $a \Leftrightarrow \hat{f}$ is smooth at $\hat{a}$.

<u>Proof</u>: By Observation B(4) $\hat{a} \in V_{\hat{f}}$ & $\frac{\partial f}{\partial x_j}(a) = \frac{\partial \hat{f}}{\partial X_j}(\hat{a}) \quad \forall j \geq 1$

($\Rightarrow$) $\exists\, j \geq 0$ such that $\frac{\partial \hat{f}}{\partial X_j}(\hat{a}) = \frac{\partial f}{\partial x_j}(a) \neq 0 \Rightarrow \hat{f}$ is smooth at $\hat{a}$

($\Leftarrow$) If f is singular at $a \Rightarrow \frac{\partial f}{\partial x_j}(a) = 0 \quad \forall j \geq 1$

& $d\hat{f}(\hat{a}) = 0 \stackrel{\text{Obs. A(2)}}{\Longrightarrow} \frac{\partial \hat{f}}{\partial X_0}(\hat{a}) = d \cdot \hat{f}(\hat{a}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) = 0 \Rightarrow \hat{f}$ is singular.