Recall that $\deg : \mathrm{Div}(L|K) \to \mathbb{Z}$ is a group homomorphism and that $\mathrm{Princ}(L|K) \overset{(6.5)}{\leq} \mathrm{Ker}(\deg) \leq \mathrm{Div}(L|K)$.

$\boxed{\text{T\&N}}$ $\mathrm{Pic}^0(L|K) := \mathrm{Ker}(\deg)/\mathrm{Princ}(L|K)$ is called the <u>Picard group</u>. Denote $[A] := A + \mathrm{Princ}(L|K) \in \mathrm{Pic}^0(L|K)$ (the coset given by A)

<u>Lemma 8.6</u> Let L be an EFF over K, $P_1, P_2, Q \in \mathbb{P}^{(1)}_{L|K}$ and $A \in \mathrm{Div}(L|K)$. Then

(1) if $P_1 - P_2 \in \mathrm{Princ}(L|K) \Rightarrow P_1 = P_2$,

(2) if $\deg A = 1 \Rightarrow \exists!\, P \in \mathbb{P}^{(1)}_{L|K} : P - A \in \mathrm{Princ}(L|K)$,

(3) The mapping $\psi_Q : \mathbb{P}^{(1)}_{L|K} \longrightarrow \mathrm{Pic}^0(L|K)$ defined by the rule $\psi_Q(P) := [P - Q]$ is a bijection.

$\boxed{\text{proof}}$: Note that $\deg A \geq 1 \overset{26(a)}{\Longrightarrow} \ell(A) = \deg(A)$

(1) ?? $\Delta \in L \setminus K \overset{(05)}{:} P_1 = P_2 + (\Delta) \overset{\text{definition}}{\Longrightarrow} \Delta, 1 \in \mathcal{L}(1P_2)$ are LI $\Rightarrow$

$\Rightarrow \ell(1P_2) = \dim_K(\mathcal{L}(1P_2)) \geq 2 > \dim_K(1P_2) = 1 \Rightarrow$ a contradiction

$\Rightarrow P_1 = P_2$

(2) $\boxed{\text{existence}}$: (a) if $A \geq 0$, $\deg A = 1 \Rightarrow \exists P \in \mathbb{P}^{(1)}_{L|K} : A = 1P$ ✓

(b) Let A is general such that $\deg A = 1 \overset{\text{defn.}}{\Longrightarrow} \exists \Delta \in L^* : \Delta \in \mathcal{L}(A)$

$\overset{\text{def.}}{\Longrightarrow} A + (\Delta) \geq 0 \overset{(a)}{\Longrightarrow} \exists P \in \mathbb{P}^{(1)}_{L|K} : A + (\Delta) = P \Rightarrow$

$P - A = (\Delta) \in \mathrm{Princ}(L|K)$

$\boxed{\text{Unicity}}$ if $A - P_1, A - P_2 \in \mathrm{Princ}(L|K)$

$\Rightarrow P_1 - P_2 \in \mathrm{Princ}(L|K) \overset{(1)}{\Longrightarrow} P_1 = P_2$

(3) $\boxed{\text{injectivity}}$ if $\varphi_Q(P_1) = \varphi_Q(P_2) \overset{(3)}{\Longrightarrow} [P_1 - P_2] = [0]$

$\Rightarrow P_1 = P_2$

surjectivity if $B \in \mathrm{Div}(L|k)$ such that $\deg B = 0 \Rightarrow$
$\deg(Q+B) = 1 \overset{(2)}{\Rightarrow} \exists!\, P \in \mathbb{P}^{(1)}_{L|k} : P - (B+Q) \in \mathrm{Princ}(L|k)$
$\Rightarrow \psi_Q(P) = [P-Q] = [B]$

T&N Let L be an EFF, $Q \in \mathbb{P}^{(1)}_{L|k}$, we define an operation $\oplus$ on $\mathbb{P}^{(1)}_{L|k}$: $P_1 \oplus P_2 := \psi_Q^{-1}(\psi_Q(P_1) + \psi_Q(P_2))$ for $P_1, P_2 \in \mathbb{P}^{(1)}_{L|k}$ and ψ_Q from 8.6(3)

<u>Observation</u> Let L be an EFF over k, $Q, P_0, P_1, \dots, P_n \in \mathbb{P}^{(1)}_{L|k}$ and ψ_Q is as in 8.6(3). Then:

(1) $\mathbb{P}^{(1)}_{L|k}$ forms an abelian group with $\oplus$ a neutral element Q,

(2) ψ_Q is a group isomorphism,

(3) $P_1 \oplus P_2 = P_3 \Leftrightarrow [P_1 + P_2] = [P_3 + Q]$,

(4) $P_1 \oplus \dots \oplus P_n = P_0 \Leftrightarrow -P_0 + (1-n)Q + \sum_{i=1}^{n} P_i \in \mathrm{Princ}(L|k)$

$\boxed{\text{T\&N}}$ Let $\hat{\ell} = cx + dy + e \in K[x,y]$ for $c,d,e \in K$, $(c,d) \neq (0,0)$.

Then $\ell = \hat{\ell} + (w) = \hat{\ell}(\alpha,\beta) \in K[V_w] = K[\alpha,\beta]$ for $\alpha = x + (w)$, $\beta = y + (w)$ is called <u>a line</u> on V_w <u>represented by</u> $\hat{\ell}$, we say that ℓ <u>passes through</u> $\gamma \in V_w$ if $\gamma \in V_{\hat{\ell}}$.

<u>Lemma 8.7</u> Let $w = y^2 + a_1xy + a_3y - (\overbrace{x^3 + a_2x^2 + a_4x + a_6}^{= f(x)})$ is a smooth WEP of $V_w(K)$, $\gamma = (\gamma_1, \gamma_2) \in V_w(K)$ and $\hat{\ell} \in K[x,y]$ represents a line $\ell = \hat{\ell} + (w) \in K[V_w]$.

(1) if $\hat{\ell} = x - \gamma_1 \Rightarrow \exists!\, \delta = (\gamma_1, \delta_2) \in V_w(K)$ such that $(\ell) = P_\gamma + P_\delta - 2P_\infty$ and $\boxed{\delta_2 = -a_1\gamma_1 - a_3 - \gamma_2}$

(2) if $\hat{\ell} = y - \lambda x - \mu$ for $\lambda, \mu \in K$ and ℓ passes through γ then $(\ell)_- = 3P_\infty$ and either:

(a) $\exists P \in \mathbb{P}_{L/K}$: $\deg P = 2$ and $(\ell)_+ = P_\gamma + P$, $\hat{\ell} \notin (\Delta_\gamma(w))$ (the tangent at γ), and $V_w(K) \cap V_{\hat{\ell}} = \{\gamma\}$

or (b) $\exists\, \delta = (\delta_1, \delta_2), \eta = (\eta_1, \eta_2) \in V_w(K) : (l)_+ = P_\gamma + P_\delta + P_\eta$,

$V_w \cap V_{\hat{l}} = \{\gamma, \delta, \eta\}$, $\eta_1 = \gamma_1 - \delta_1 - a_2 + \lambda^2 + a_1\lambda$, and

$\hat{l} \in (A_\gamma(w)) \Leftrightarrow \gamma \in \{\delta, \eta\}$

<u>proof</u>: Recall: $\alpha = x + (w)$, $\beta = y + (w)$, $l = \hat{l}(\alpha, \beta) \in K[V_w] = K[\alpha, \beta]$
and $K(\alpha, \beta) = L$.

(1) By 8.3: $(l)_- = (\alpha - \gamma_1)_- \overset{(v_P(\gamma_1) = 0\ \forall P)}{=\!=\!=} (\alpha)_- \overset{8.3(3)}{=} 2P_\infty$,

by 5.8 $P_\gamma \le (l)_+$ as $\hat{l}(\gamma) = 0$, by 8.3(4): $\deg P_\gamma = 1$

$\overset{6.5}{\Longrightarrow} \exists!\, P \in \mathbb{P}^{(1)}_{L|K} : (l)_+ = P_\gamma + P \overset{8.3(4)}{\Longrightarrow} \exists!\, \delta = (\delta_1, \delta_2) \in V_w(K)$:
$P = P_\delta \Rightarrow \boxed{(l)_+ = P_\gamma + P_\delta} \Rightarrow$

$\Rightarrow P_\delta \le (l)_+ \overset{5.8}{\Longrightarrow} \delta \in V_{\hat{l}} \Rightarrow \boxed{\delta_1 = \gamma_1}$

Since $w(\gamma) = 0 = w(\delta)$ $\exists\, \lambda \in K$ such that δ_2, γ_2 are roots of $s^2 + \underline{a_1\gamma_1}\, s + \underline{a_3}\, s + \lambda \in K[s] \Rightarrow \boxed{\delta_2 + \gamma_2 = -a_1\gamma_1 - a_3}$

(2) Again by 8.3(4) & 5.8 : $(\ell)_- = 3P_\infty$ and $P_\gamma \leq (\ell)_+$

Then by 6.5

– either (a): $(\ell)_+ = P_\gamma + P$ for $P \in \mathbb{P}_{L|K}$ of degree 2 $\Rightarrow$

$\Rightarrow P_\delta \nleq (\ell)_+ \ \forall \delta \in V_w(K) \setminus \{\gamma\} \Rightarrow V_w(K) \cap V_{\hat{\ell}} = \{\gamma\}$

– or (b) $(\ell)_+ = P_\gamma + P_\delta + P_\eta$ for some $\delta = (\delta_1, \delta_2)$, $\eta = (\eta_1, \eta_2) \in V_w(K)$

Note: $\hat{\ell} \in (A_\gamma(w)) \overset{5.8}{\iff} 2P_\gamma \leq (\ell)_+ \iff \gamma \in \{\delta, \gamma\}$

(which is impossible in the case (a))

Let $w = y^2 + a_1xy + a_3y - f(x)$ with $f(x) = x^3 + a_2x^2 + a_4x + a_6$

Suppose (b): $(\ell)_+ = P_\gamma + P_\delta + P_\eta$ and put

$$\underbrace{g(A) = -w(A, \lambda A + \mu)}_{\in K[A]} = f(A) - (\lambda A + \mu)^2 - a_1(\lambda A + \mu) - a_3(\lambda A + \mu) \Rightarrow \boxed{\deg g = 3}$$

if $\rho \in \overbrace{\{\gamma_1, \delta_1, \eta_1\}}^{R :=}$ or $(\rho, \tau) \in V_w \cap V_{\hat{\ell}} \Rightarrow g(\rho) = 0.$

if $|R| = 3$ or $\quad$ (if $|R| < 3 \Rightarrow \hat{\ell} \in (\Delta_\xi(w))$ for some $\xi \in R \Rightarrow \frac{\partial w}{\partial \xi}(\xi) \neq 0$)

by 8.9 $\Rightarrow$ The multiplicity of the root $\varrho \in \{\gamma_1, \delta_1, \eta_1\}$ is equal to $v_P(\ell)$ for the corresponding $P \in \{P_\gamma, P_\delta, P_\eta\}$

$\Rightarrow$ $\{\gamma_1, \delta_1, \eta_1\}$ are exactly all roots of the monic polynomial $g(\lambda)$

$\Rightarrow$ the coefficient of λ^2 of g is $\boxed{-(\alpha_1 + \beta_1 + \gamma_1) = a_2 - \lambda^2 - a_1 \lambda}$

<u>Definition</u>: Let w be a smooth WEP and L be an EFF (given by $w(\alpha, \beta) = 0$). Consider the group structure on $\mathbb{P}^{(1)}_{L|K}$ determined by Ψ_{P_∞} from 8.6(3). Put $E(K) := V_w(K) \cup \{\infty\}$ and define operations $\oplus$ and $\ominus$ on $E(K)$; $\gamma, \delta, \eta \in E(K)$:

$$\boxed{\gamma \oplus \delta = \eta} \overset{\text{def}}{\equiv} P_\gamma \oplus P_\delta = P_\eta \overset{\text{Ob. (3)}}{\Longleftrightarrow} [P_\gamma + P_\delta] = [P_\eta + P_\infty]$$

$$\boxed{\ominus \gamma = \delta} \overset{\text{def}}{\equiv} P_\gamma \oplus P_\delta = P_\infty$$