Lemma 6.9 Let $K=\tilde{K}$ and $A \in Div(L|K)$: $\deg A = 0$; then (1) $l(A) \in \{0,1\}$

(2) $l(A) = 1 \Leftrightarrow A \in Prime(L|K)$

Proof: Let $l(A) \geq 1 \overset{(D7)}{\Longrightarrow} \exists \Delta \in L^* : A + (\Delta) \geq 0 \overset{6.5}{\Longrightarrow}$
$\Rightarrow \deg(\underbrace{A+(\Delta)}_{\geq \underline{0}}) = 0 + 0 = 0 \Rightarrow A + (\Delta) = \underline{0} \Rightarrow A = (\Delta^{-1})$
$\overset{(D9)}{\Longrightarrow} l(A) = \dim_K \Delta^{-1}K = 1$, the rest is clear

Theorem 6.10 (Riemann) If $K = \tilde{K}$, then $\exists \gamma \in \mathbb{Z}$ such that $\gamma \geq 0$, $\deg A - l(A) < \gamma \;\; \forall A \in Div(L|K)$.

Proof: Since $A \leq A_+ \overset{(D4)}{\Longrightarrow} \deg A - l(A) \leq \deg A_+ - l(A_+)$
So it is enough to prove the claim for $A \geq \underline{0}$
Let $A \geq \underline{0}$ and $\Delta \in L \setminus K$, denote $\boxed{C := (\Delta)_-}$

Then $rC = (s^r)_- \neq 0 \ \forall r \geq 0 \Rightarrow r \deg C > 0$

6.5 & 6.4(3) $\Rightarrow \deg(rC) - \ell(rC) = r \cdot [L:K(s)] - \ell(rC)$

$\exists B \geq \underline{0}$ $\qquad \leq \deg B - [L:K(s)]$

Put $\gamma := \deg B - [L:K(s)] + 1 \Rightarrow$ $\boxed{\deg(rC) - \ell(rC) < \gamma}$

Since $A \geq \underline{0} \Rightarrow rC - A \leq rC \overset{(D4)}{\Rightarrow}$

$\Rightarrow \underline{\deg(rC-A) - \ell(rC-A)} \leq \deg(rC) - \ell(rC) \underline{< \gamma}$

$\Rightarrow \ell(rC - A) > -\gamma - \deg A + r \underbrace{\deg C}_{>0} \quad \forall r > 0$

$\exists r_0 : \forall r \geq r_0 \ \ell(rC - A) \geq 1 \overset{(D6)}{\Rightarrow}$

$\Rightarrow \underline{\deg A - \ell(A)} \leq \deg(rC) - \ell(rC) \underline{< \gamma}$

<u>Definition</u>: The minimal γ from 6.10 for an AFF L over $\bar{K}$ (i.e. $\deg A - \ell(A) < \gamma \ \forall A \in \mathrm{Div}(L/\bar{K})$) is called <u>genus</u> <u>of L over K</u>, it will be denoted by $\boxed{g}$.

Observation (D) Let $A, D \in \mathrm{Div}(L|K)$, $K = \tilde{K}$ and $\deg D - \ell(D) = g - 1$ for the genus g. Then

(1) $g > \deg 0 - \ell(0) = -1 \Rightarrow g \geq 0$

(2) $\deg(A-D) - \ell(A-D) \leq g - 1$ (by the definition)

$$\Rightarrow \ell(A-D) \geq \deg A - \deg D - g + 1$$

(3) if $\deg A \geq \deg D + g \overset{(2)}{\Longrightarrow} \ell(A-D) \geq 1$

(4) if $\ell(A-D) \geq 1$ or $D \leq A \overset{(D4)\&(D6)}{\Longrightarrow}$

$$\Rightarrow g - 1 = \deg D - \ell(D) \leq \deg A - \ell(A) \leq g - 1 \Rightarrow$$

$$\Rightarrow \deg A - \ell(A) = g - 1$$

Lemma 6.11 $\exists \gamma \in \mathbb{N}$ such that $\forall A \in \mathrm{Div}(L|K)$ with $\deg(A) \geq \gamma$: $\deg(A) = \ell(A) + g - 1$

Proof of 6.11: it is enough to put $\gamma := \deg D + g$ from Obs.(D) and to apply Obs. D (B), (C)

T 8.N Let $\mathbb{P} := \mathbb{P}_{L|K}$ and consider $L^{\mathbb{P}}$ as an L-algebra with the operations $(f \pm g)(P) = f(P) \pm g(P)$, $\underline{0}(P) = 0$, $\underline{1}(P) = 1$ $\forall P \in \mathbb{P}$ where $l \to l \cdot \underline{1}$ identifies L and $\{l \cdot \underline{1} \mid l \in L\}$.
$f \in L^{\mathbb{P}}$ is called an adèle if $|\{P \in \mathbb{P} \mid f(P) \notin \mathcal{O}_P\}| < \infty$ and $\mathcal{A}_{L|K}$ denotes the set of all adèles (over L/K).
Let $A = \sum a_P P \in Div(L/K)$, then
$\mathcal{A}_{L|K}(A) := \{f \in L^{\mathbb{P}} \mid v_P(f(P)) + a_P \geq 0 \ \forall P \in \mathbb{P}\}$ and
$i(A) := g - 1 - \deg A + \ell(A) \geq 0$ is the index of speciality. A is called < special if $i(A) > 0$; nonspecial if $i(A) = 0$

Recall that $P^r = \{ r \in L \mid v_P(r) \geq r \}$ $\forall r \in \mathbb{Z}, \forall P \in \mathbb{P}_{L|K}$

<u>Observation (E)</u> Let $\mathbb{P} := \mathbb{P}_{L|K}$, $r \in L$, $f \in L^{\mathbb{P}}$,

$$A = \sum_{P \in \mathbb{P}} a_P P \in Div(L|K), \text{ then:}$$

(1) $f \in \mathcal{A}_{L|K} \iff v_P(f(P)) < 0$ for only finitely many $P \in \mathbb{P}$

so $r = r \cdot \underline{1} \in \mathcal{A}_{L|K}$ by 5.22,

(2) $\mathcal{A}_{L|K}$ is a L-subalgebra of the L-algebra $L^{\mathbb{P}}$,

(3) $f \in \mathcal{A}_{L|K}(A) \iff v_P(f(P)) \geq -a_P$ $\forall P \in \mathbb{P}$,

(4) $\mathcal{A}_{L|K}(A) = \prod_{P \in \mathbb{P}} P^{-a_P}$ is a K-subspace of $\mathcal{A}_{L|K}$

and $\mathcal{A}_{L|K}(A) \cap L = \mathcal{L}(A)$,

(5) $\mathcal{A}_{L|K} = \bigcup_{B \in Div(L|K)} \mathcal{A}_{L|K}(B)$.

<u>Lemma 6.12</u> Let $K = \bar{K}$, $A = \sum a_P P$, $B = \sum b_P P \in \mathrm{Div}(L|K)$, $\Delta \in L^*$:

(1) if $A \leq B \Rightarrow \mathcal{A}_{L|K}(A) \subseteq \mathcal{A}_{L|K}(B)$ and
$$\dim_K(\mathcal{A}_{L|K}(B)/\mathcal{A}_{L|K}(A)) = \deg(B-A),$$

(2) if $A \leq B \Rightarrow \dim_K((\mathcal{A}_{L|K}(B)+L)/(\mathcal{A}_{L|K}(A)+L)) = i(A) - i(B)$,

(3) $\mathcal{A}_{L|K}(A) \cap \mathcal{A}_{L|K}(B) = \mathcal{A}_{L|K}(\min(A,B))$,
$\mathcal{A}_{L|K}(A) + \mathcal{A}_{L|K}(B) = \mathcal{A}_{L|K}(\max(A,B))$,

(4) $\dim_K(\mathcal{A}_{L|K}/(\mathcal{A}_{L|K}(A)+L)) = i(A)$,

(5) $\mathcal{A}_{L|K}(A) + L = \mathcal{A}_{L|K} \iff i(A) = 0$,

(6) $\Delta \cdot \mathcal{A}_{L|K}(A) = \mathcal{A}_{L|K}(A - (\Delta))$

<u>Proof</u>: (1) $A \leq B \Rightarrow \mathcal{A}_{L|K}(A) = \{f \mid f(P) \geq -a_P (\geq -b_P)\ \forall P\} \subseteq \mathcal{A}_{L|K}(B)$

$$\deg(B-A) \underset{\uparrow}{=} \dim_k\left(\prod_P P^{-b_P} / \prod_P P^{-a_P}\right) \underset{\uparrow}{=} \dim_k \left(\mathcal{A}_{L|k}(B) / \mathcal{A}_{L|k}(A)\right)$$

by Observ. C(3) using the argument of the proof of 6.2 — Observation E(4)

(2) $A \le B \Rightarrow \dim_k\left((\mathcal{A}_{L|k}(B)+L)/(\mathcal{A}_{L|k}(A)+L)\right) \overset{1.3(2)}{=\!=}$

$= \dim\left(\mathcal{A}_{L|k}(B)/\mathcal{A}_{L|k}(A)\right) - \underbrace{\dim\left(\mathcal{A}_{L|k}(B)\cap L\right)/\left(\mathcal{A}_{L|k}(A)\cap L\right)}_{=\mathcal{L}(B)/\mathcal{L}(A) \text{ by } E(4)}$

$\overset{(1)}{=} \deg B - \deg A - (\ell(B)-\ell(A)) = i(A) - i(B).$

(3) it follows by Obs. E(4) & : $\prod P^{-a_P} + \prod P^{-b_P} = \prod P^{\min(-a_P,-b_P)}$ $= -\max(a_P, b_P)$

— " — $\cap$ — " — $\prod P^{\max(-a_P,-b_P)}$, $= -\min(a_P, b_P)$

(4) (a) First suppose $i(A)=0$ and we will show $\mathcal{A}_{L|k} = \mathcal{A}_{L|k}(A)+L$ $\boxed{\supseteq}$ – follows by Obs E(1),(5)

$\boxed{\subseteq}$ let $f \in \mathcal{A}_{L|k}$, put $\boxed{d_P := \max(a_P, 0, -v_P(f(P)))}$ $\forall P \in \mathbb{P}_{L|k}$

and $D := \sum d_P P \Rightarrow d_P \ge a_P,\ d_P + v_P(f(P)) \ge 0\ \forall P \Rightarrow \underline{A \le D,\ f \in \mathcal{A}_{L|k}(D)}$

$\overset{\text{Obs. } D(4)}{\underset{i(A)=0}{\Longrightarrow}} i(D)=0 \overset{(2)}{\Longrightarrow} f \in \mathcal{A}_{L|K}(D)+L = \mathcal{A}_{L|K}(A)+L$

(b) let A be general, then $\exists B \geq A : i(B)=0$ by 6.11

$\overset{(a)}{\Longrightarrow} \mathcal{A}_{L|K} = \mathcal{A}_{L|K}(B)+L \Rightarrow$

$\underline{\dim_k(\mathcal{A}_{L|K}/(\mathcal{A}_{L|K}(A)+L))} = \dim_k((\mathcal{A}_{L|K}(B)+L)/\mathcal{A}_{L|K}(A)+L) \overset{(2)}{=} i(A) - \underbrace{i(B)}_{0} = \underline{i(A)},$

(5) it is a particular case of (4)

(6) Let $f \in \mathcal{A}_{L|K}$: $\boxed{f \in \Delta \mathcal{A}_{L|K}(A)} \Leftrightarrow \Delta^{-1} f \in \mathcal{A}_{L|K}(A) \Leftrightarrow$

$\Leftrightarrow v_P(\Delta^{-1} f(P)) + a_P \geq 0 \ \forall P \Leftrightarrow v_P(f(P)) + a_P - v_P(\Delta) \geq 0 \ \forall P \Leftrightarrow \underline{f \in \mathcal{A}_{LK}(A-(\Delta))}$