$\boxed{L \text{ is an AFF over } K,\ \tilde{K} \text{ - a field of constants}}$

<u>Errata</u>: $A \in \mathrm{Div}(L|K)$: $\boxed{A_- := -\min(A, \underline{0})} = (-A)_+ = A_+ - A$

<u>Lemma 6.2</u> If $A, B \in \mathrm{Div}(L|K)$ such that $A \leq B$, then $\mathcal{L}(A)$ is a subspace of $\mathcal{L}(B)$, $\dim_K(\mathcal{L}(B)/\mathcal{L}(A)) \leq \deg_{K}(B-A)$.

<u>proof</u>: Let $A = \sum_{P \in \mathbb{P}_{L|K}} a_P P$, $B = \sum_{P \in \mathbb{P}_{L|K}} b_P P$

$r \in \mathcal{L}(A) \Rightarrow (r) + B \underset{\text{Obs. } B(4)}{\geq} (r) + A \geq 0 \Rightarrow r \in \mathcal{L}(B) \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(B)$

Remind: $r \in \mathcal{L}(B) \Leftrightarrow (r) \geq -B \Leftrightarrow v_P(r) \geq -b_P \ \forall P \in \mathbb{P}_{L|K}$

$r \in \mathcal{L}(A) \Leftrightarrow v_P(r) \geq -a_P \ \forall P \in \mathbb{P}_{L|K}$

Define $\varphi: \mathcal{L}(B) \to \prod_{P \in \mathbb{P}_{L|K}} P^{-b_P} := \prod p^{-b_P} \mathcal{O}_P$ where $(p) = P$ (p generates P)

$0 \neq r \to \prod_P r$ (the constant in all coordinates), $\varphi(\underline{0}) := \prod 0$

Note that $\varphi(r) \in \prod_P P^{-a_P} \Leftrightarrow r \in \mathcal{L}(A)$, φ is K-homomorphism

By 1.2 φ induces embedding $\mathcal{L}(B)/\mathcal{L}(A) \hookrightarrow \frac{\prod_P P^{-b_P}}{\prod_P P^{-a_P}} \cong \prod_{P \in \mathbb{P}_{L|K}} P^{-b_P}/P^{-a_P}$

$\dim_K P^{-b_P}/P^{-a_P} \overset{\boxed{\text{Rem. C(3)}}}{=} (b_P - a_P)\deg P \geq 0 \Rightarrow \dim \mathcal{L}(B)/\mathcal{L}(A) \leq \sum (b_P - a_P) \deg P = \deg(B-A)$

Proposition 6.3. Let $K = \tilde{K}$ (i.e. L is "full constant AFF"), $A, B \in \mathrm{Div}(L|K)$

(D1) if $A \geq \underline{0} \Rightarrow 1 \leq \ell(A) \leq \deg A + 1$

(D2) if $A < \underline{0} \Rightarrow \ell(a) = 0$

(D3) $\ell(A) \leq \ell(A_+) < \infty$

(D4) if $A \leq B \Rightarrow \deg A - \ell(A) \leq \deg B - \ell(B)$

proof: (D1) $0 \leq \dim(\mathcal{L}(A)/\mathcal{L}(\underline{0})) = \ell(A) - \ell(\underline{0}) \overset{\text{Obs. B(6)}}{=} \ell(A) - 1$
$\Rightarrow \ell(A) \geq 1$

by 6.2. $\ell(A) - 1 = \dim(\mathcal{L}(A)/\mathcal{L}(\underline{0})) \leq \deg A - \deg \underline{0} = \deg A$
$\Rightarrow \ell(A) \leq \deg A + 1$

(D2) $A < 0 \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(\underline{0}) = \tilde{K} = K$

if $\lambda \in K^* \Rightarrow v_P(\lambda) = 0 \ \forall P \Rightarrow (\lambda) = \underline{0} \Rightarrow A + (\lambda) < 0 \Rightarrow \mathcal{L}(A) = \{0\}$

(D3) $\mathcal{L}(A) \subseteq \mathcal{L}(A^+) \Rightarrow \underbrace{\ell(A) \leq}_{\ell(B)} \ell(A_+) \underset{\ell(A)}{\underline{<\infty}}$ by (D1)

(D4) (D3) & Lemma 6.2 $\Rightarrow \overbrace{\dim \mathcal{L}(B)}^{=\ell(B)} - \overbrace{\dim \mathcal{L}(A)}^{\ell(A)} \leq \deg B - \deg A$

finite

<u>Lemma 6.4</u> If $\Delta \in L \setminus \tilde{K}$, then $\exists B \in \mathrm{Div}(L|K)$ such that $B \geq \underline{0}$ and $\forall r \geq 0$:

(1) $(r+1)\,[L : K(\Delta)] \leq \ell(r \cdot (\Delta)_- + B)$

(2) —— " —— $\leq r \cdot \deg_K((\Delta)_-) + \deg_K B + 1$

(3) $r \cdot [L : K(\Delta)] - \ell(r \cdot (\Delta)_-) \leq \deg_K B - [L : K(\Delta)]$

Proof: (1) Let $e_1, \dots, e_n \in L$ be a $K(\Delta)$-basis of L

$\overset{1.9}{\Longrightarrow} E_r := \{e_i \Delta^j \mid i = 1, \dots, n;\ j = 0, \dots, r\}$ lin. indep. over K (as $K(\Delta) \cong K(x)$)

we find $B = \sum b_P P$ such that $E_r \subseteq \mathcal{L}(r \cdot (\Delta)_- + B)$ $\forall r \geq 0$

Define $\boxed{b_P := \max\{0, -v_P(e_1), \dots, -v_P(e_n)\} \ \forall P} \Rightarrow b_P + v_P(e_i) \geq 0 \ \forall P$ & $b_P \geq 0$

$B := \sum_{P \in \mathbb{P}_{L|K}} b_P P$ is correctly defined $\in \mathrm{Div}(L|K)$ since

$\{P \in \mathbb{P}_{L|K} \mid b_P \neq 0\} \subseteq \bigcup_{i=1}^{n} \{P \in \mathbb{P}_{L|K} \mid v_P(e_i) \neq 0\}$ which is finite by 5.22

Recall $e_i s^j \in \mathcal{L}(r\cdot(s)_- + B) \iff D := B + (e_i s^j) + r(s)_- \geq 0$ $\quad$ $\sum d_P P$

we will check coefficients d_P of D; let $P \in \mathbb{P}_{L|K}$:

if $v_P(s) < 0 \Rightarrow \underline{d_P} = \overbrace{b_P + v_P(e_i)}^{\geq 0} + j v_P(s) - r v_P(s) \geq \overbrace{(j-r)}^{\leq 0} \overbrace{v_P(s)}^{<0} \underline{\geq 0}$

if $v_P(s) \geq 0 \Rightarrow \underline{d_P} = \overbrace{b_P + v_P(e_i)}^{\geq 0} + j v_P(s) \geq j v_P(s) \underline{\geq 0}$

$\Rightarrow \ell(r(s)_- + B) \geq |E_r| = (r+1)[L : K(s)]$

(2) $\underbrace{(r+1)[L:K(s)]}$ $\overset{(1)}{\leq} \ell(\underbrace{\overbrace{r\cdot(s)_-}^{\geq 0} + \overbrace{B}^{\geq 0}}_{\geq 0}) \leq \deg(r(s)_- + B) + 1 = \boxed{r \cdot \deg((s_-)) + \deg B + 1}$ by 6.5(b1)

(3) $B \geq 0$ $\overset{\text{Obs } B(+)}{\Rightarrow}$ $r\cdot(s)_- \leq r(s)_- + B \overset{6.2}{\Rightarrow} \ell(r(s)_- + B) - \ell(r(s)_-) \leq$ $\deg(\text{—''—}) - \deg(\text{—''—})$

$\overset{(1)}{\Rightarrow} \underbrace{(r+1)[L:K(s)] - \ell(r(s)_-)} \leq \deg(r(s)_- + B) - \deg(r(s)_-) = \underline{\deg B}$

Theorem 6.5 Let $K = \tilde{K}$ and $s \in L \setminus \tilde{K}$. Then

$$\deg((s)_-) = \deg((s)_+) = [L : K(s)] \text{ and } \deg((s)) = 0.$$

Proof: (a) $\deg((s)_+) = \sum_{P : s \in P} v_P(s) \deg P \leq [L : K(s)]$ by 5.21

(b) $K(s) = K(s^{-1})$ & $(s)_- = (s^{-1})_+ \Rightarrow$

$$\deg((s)_-) = \deg((s^{-1})_+) \overset{(a)}{\leq} [L : K(s^{-1})] = [L : K(s)]$$

(c) $6.4(2) \Rightarrow \exists B \in \mathrm{Div}(L|K) : \forall r \geq 0$

$(r+1)[L : K(s)] \leq r \cdot \deg((s)_-) + \deg B + 1 \Rightarrow$

$\Rightarrow [L : K(s)] \leq \deg((s)_-) + \frac{-[L:K(s)] + \deg B + 1}{r} \to \deg((s)_-)$ for $r \to \infty$

this & (a) $\Rightarrow [L : K(s)] = \deg((s)_-)$

(d) (b) & (c) $\Rightarrow \deg((s)_+) = \deg((s^{-1})_-) = [L : K(s^{-1})] = [L : K(s)]$

finally: $\deg((s)) = \deg((s)_+ - (s)_-) = \deg((s)_+) - \deg((s)_-) = 0$

Corollary 6.6 If $A \sim B$, then (1) $\deg A = \deg B$
(2) $\dim_{L|k} A = \dim_{L|k} B$

Proof: $A \sim B \Leftrightarrow \exists\, \lambda \in L^* : A = B + (\lambda)$

(1) $\deg A = \deg(B+(\lambda)) = \deg B + \overbrace{\deg((\lambda))}^{=0 \text{ by } 6.5} = \deg B$

(2) $\mathcal{L}(A) \to \mathcal{L}(B)$ is bijective as $\underline{r \in \mathcal{L}(A)} \Leftrightarrow (r) + \overset{B+(\lambda)}{\overset{\shortparallel}{A}} \geq 0 \Leftrightarrow \underline{(\lambda \cdot r) + B \geq 0}$,
$r \to \lambda \cdot r$ & it is $\bar{k}$-linear (and so k-linear)

Example 6.7 Consider L over $\mathbb{F}_2$ given by $f(\alpha,\beta) = 0$ from Ex. 5.24, i.e. $f = y^2 + y - (x^3+1) \in \mathbb{F}_2[x]$

(a) $\deg((\alpha+1)_+) = \sum_{P:\, \alpha+1 \in P} v_P(\alpha+1) \deg P \overset{6.5}{=} [L : \overbrace{\mathbb{F}_2(\alpha+1)}^{=\mathbb{F}_2(\alpha)}] = 2$, $\alpha+1 \in P_{(1,0)}, P_{(1,1)} \in \mathbb{P}_{L|\mathbb{F}_2}$
$\Rightarrow \deg P_{(1,0)} = \deg P_{(1,1)} = 1 \quad \Rightarrow \boxed{\{P_{(1,0)}, P_{(1,1)}, P_\infty\}}$ is the set of all places of degree 1
$\Rightarrow \boxed{(\alpha+1) = 1 \cdot P_{(1,0)} + 1 \cdot P_{(1,1)} - 2 \cdot P_\infty \in \mathrm{Div}(L/\mathbb{F}_2)}$

(b) $\deg((\alpha)_+) = \sum_{P:\, \alpha \in P} v_P(\alpha) \deg P \underset{6.5}{=} 2$, $\alpha \notin$ place of degree 1 $\Rightarrow \exists!\, P_\alpha \in \mathbb{P}_{L|\mathbb{F}_2} : \alpha \in P_\alpha$
$\deg P_\alpha = 2$ and $\boxed{(\alpha) = 1 \cdot P_\alpha - 2 \cdot P_\infty}$

Proposition 6.8 Let $k=\tilde{k}$, $A, B\in Div(L|k)$, then:

(D5) $\ell(B-A)\geq 1 \iff \exists A'\in Div(L|k): A\sim A'\leq B$

(D6) if $\ell(B-A)\geq 1 \Rightarrow \deg A-\ell(A)\leq \deg B-\ell(B)$,

(D7) $\ell(A)\geq 1 \iff \exists \Delta\in L^*: A+(\Delta)\geq 0$

(D8) if $\deg A<0 \Rightarrow \ell(a)=0$

(D9) $\mathcal{L}((\Delta))=k\Delta^{-1}=\{z\Delta^{-1} \mid z\in k\}\ \forall \Delta\in L^*$

Proof: (D5) $\ell(B-A)\geq 1 \iff \exists \Delta\in L^*: (\Delta)+B-A\geq 0 \iff$

$\iff \exists \Delta\in L^*: B\geq A-(\Delta) \overset{\text{defin}}{\iff} \exists \underset{=A-(\Delta)}{A'}\sim A: A'\leq B$

(D6) by (D5) $\exists A'\sim A: A'\leq B \xRightarrow{6.3(D4)} \underbrace{\deg A'-\ell(A')}_{\overset{6.6}{=}\deg A-\ell(A)}\leq \deg B-\ell(B)$

(D7) $\ell(A)\geq 1 \iff$

$\iff \exists \Delta\in \mathcal{L}(A)-\{0\} \iff \exists \Delta\in L^*: A+(\Delta)\geq 0$

(D8) $\deg A < 0 \Rightarrow \deg(A + (s)) \overset{6.5}{=} \deg(A) < 0 \quad \forall s \in L^*$ [8]

$\overset{(\text{def.})}{\Longrightarrow} \mathcal{L}(A) = 0$

(D9) $r \in \mathcal{L}((s)) \Leftrightarrow (r \cdot s) = (r) + (s) \geq 0 \overset{\text{Obs. B(6)}}{\Leftrightarrow} r \cdot s \in K$

$\Leftrightarrow r \in K s^{-1}$