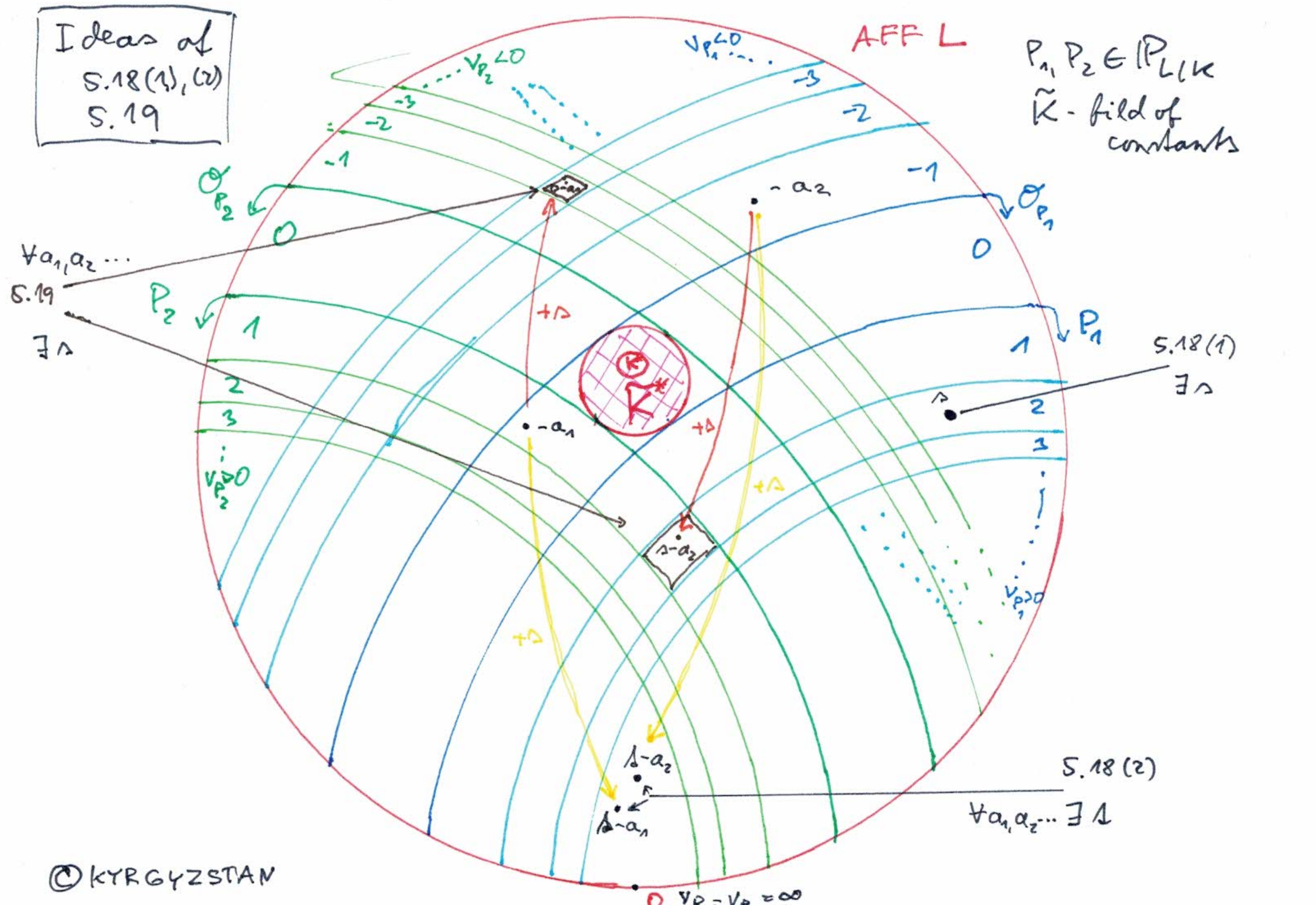

Ideas of
5.18(1), (2)
5.19
AFF L
P1, P2 ∈ PL|K
K̃ - fild of constants
vP1 < 0
vP2 < 0
-3
-2
-1
0
1
2
3
OP2
OP1
P2
P1
vP2 > 0
vP1 > 0
∀a1, a2 ...
5.19
∃ s
5.18(1)
∃ s
s
-a2
-a1
+s
s-a1
s-a2
K̃
K̃*
5.18 (2)
∀a1, a2 ... ∃ s
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proof of Lemma 8.18: (1) we have $P_1, \dots, P_n \in \mathbb{P}_{L|K}$

$V_i := V_{P_i}$ and we will prove that $\boxed{\exists\, \Delta \in L^* \text{ such that } V_1(\Delta) > 0 \text{ and } V_i(\Delta) < 0 \ \forall i > 1}$

by induction on n: (a) $n = 1$: $\Delta \in P_1 \Rightarrow V_1(\Delta) > 0$ ✓

(b) let $n = 2$, then $\mathcal{O}_1$ & $\mathcal{O}_2$ are distinct VR $\Rightarrow$

$\Rightarrow a \in \mathcal{O}_1 \setminus \mathcal{O}_2,\ b \in \mathcal{O}_2 \setminus \mathcal{O}_1 \Rightarrow V_1(a) \geq 0,\ V_1(b) < 0,$
$V_2(a) < 0,\ V_2(b) \geq 0$

Thus $V_1\left(\frac{a}{b}\right) = V_1(a) - V_1(b) > 0,\ V_2\left(\frac{b}{a}\right) = V_2(a) - V_2(b) < 0$

c) let $n \geq 2$ and $\exists \tilde{\Delta}: V_1(\tilde{\Delta}) > 0,\ V_i(\tilde{\Delta}) < 0\ \forall i = 2, \dots, n$

? $n+1$: if $V_{n+1}(\tilde{\Delta}) < 0$ then $\Delta := \tilde{\Delta}$ and we are done

if $V_{n+1}(\tilde{\Delta}) \geq 0 \xRightarrow{(b)} \exists\, r: V_1(r) > 0\ \&\ V_{n+1}(r) < 0 \xRightarrow{\boxed{\text{Obs. (3)}}}$

$V_1(\Delta + r^k) \overset{DZV}{\geq} \min(\underbrace{V_1(\Delta)}_{>0}, \underbrace{k V_1(r)}_{>0}) > 0$; $\exists k$ large enough: $V_i(\Delta + r^k) < 0$ for $i \geq 2$

(2) we have again $P_1, \dots, P_n \in \mathbb{P}_{L|K}$, $v_i := v_{P_i}$, $k \in \mathbb{Z}$ and $a_1, \dots, a_n \in L$ and we show $\boxed{\text{that } \exists\, s \in L :\ v_i(s - a_i) > k \ \forall i = 1..n}$

we put $r_1 := (1+s^k)^{-1}$ where $s \in L^*$ (it cannot be $= 0$) : $v_1(s) > 0$, $v_i(s) < 0$ $\forall i \geq 2$ which exists by (1)

Then $s \in L - \tilde{K}$ by Obs. (2), $\boxed{\underset{\in O_1^*}{v_1(r_1 - 1)} = v_1\left(\frac{s^k}{1+s^k}\right) = k v_1(s) - \underbrace{v_1(1+s^k)}_{=0} = k v_1(s) \geq \cancel{k}}$

and $v_i(r_1) \overset{\geq 0}{=} -k \underbrace{v_i(s)}_{<0} \geq k$ $\forall i \geq 2$

By the same way we can define $r_2, \dots, r_n$:

$\forall i\ \forall j \neq i\ v_i(r_j) \geq k$ & $v_i(r_i - 1) \geq k$

Let $\boxed{s := \sum a_i r_i}$ and $\boxed{b_j := s - a_j}$

for $i \neq j$: $V_i(a_j r_j) = V_i(a_j) + \underbrace{V_i(r_j)}_{\geq k} > k + V_i(a_j)$

$V_i(a_i(r_i - 1)) = V_i(a_i) + \underbrace{V_i(r_i + 1)}_{\geq k} > k + V_i(a_i)$

(strictly greater)

$\Rightarrow$ (if $k \geq M - V_i(a_j)$ $\forall j \forall i$) $\Rightarrow$

$V_i(b_i) = V_i(\Delta - a_i) = V_i\left(\sum_{j \neq i} a_j r_j + a_i(r_i - 1)\right) > M$

$\boxed{\text{T\&N}}$ If W is a subspace of a K-space V, $B \subset V$, we say that B is LI / a basis modulo W iff $\{b + W \mid b \in B\}$ forms a LI set / basis of V/W.

Now we formulate consequences of 5.19:

<u>Corollary 5.20</u> (1) $\mathbb{P}_{L|K}$ is infinite

(2) If $P, P_1, \dots, P_n \in \mathbb{P}_{L|K}$ are pairwise distinct and $l \geq 0$, then $\exists$ a basis B of the K-algebra $\mathcal{O}_P$ modulo P such that $B \subseteq P_j^l \; \forall j$ (i.e. $v_{P_j}(b) \geq l$ $\forall b \in B$).

<u>proof</u>: (1) ?? $\mathbb{P}_{L|K} = \{P_1, \dots, P_n\} \overset{5.19}{\Longrightarrow} \exists\, s \in L^*$: $v_{P_i}(s) = 1 \;\forall i \Rightarrow s$ is transcendental over $K \Rightarrow$
$\Rightarrow K[s^{-1}] \subsetneqq \mathcal{R}L$, $(s^{-1}) \subsetneqq K[s^{-1}] \overset{2.5}{\Longrightarrow} \exists\, Q \in \mathbb{P}_{L|K} : s^{-1} \in Q$
$\Rightarrow v_Q(s) = -1 \Rightarrow$ a contradiction

(2) Put $d := \deg P = \dim_K(\mathcal{O}_P/P)$

Let $\{e_1, \dots, e_d\}$ be a K-basis of $\mathcal{O}_P$ modulo P

We use 8.19 n times for $a_0 = 0$ and $a_j = e_i$ $\forall j, \forall i$

$\Rightarrow \forall i = 1 \ldots n \;\exists s_i \in L : \; v_{P_j}(s_i - e_i) = e \;\; \forall j \geq 1$

$v_P(s_i) = 1$

Then $b_i = s_i - e_i \in P_j^e$ $\forall j \geq 1$ and $s_i \in P \Rightarrow$

$\{b_1 + P, \ldots, b_n + P\} = \{-e_1 + P, \ldots, -e_n + P\}$ is a basis of $\mathcal{O}_P/P$

<u>Observation</u>: Let $P \in \mathbb{P}_{L|K}$; $b_1, \ldots, b_n \in \mathcal{O}_P$ be LI modulo P over K, $s \in P$, $v_P(s) = 1$, $\lambda_i, \lambda_{ij} \in K$ $\forall i = 1..n, j = 0, \ldots e-1$ and $\lambda_i \neq 0$ and $\lambda_{ij} \neq 0$ for at least one i, $e \in \mathbb{N}$

(1) $v_P(\sum_i \lambda_i b_i) = 0$;

(2) $v_P(\sum_i \lambda_i b_i s^j) = v_P(\sum \lambda_i b_i) + v_P(s^j) = j$ by (1)

(3) $v_P(\sum_{ij} \lambda_{ij} b_i s^j) = \min\{j \mid \exists i : \lambda_{ij} \neq 0\}$ by 2.13 & (2)

(4) the set $\{b_i s^j \mid i = 1..n, j = 0, \ldots e-1\}$ is LI modulo P^e over K (by (3))

Proposition 5.21 Let $P_1, \dots, P_n \in \mathbb{P}_{L|K}$ be pairwise distinct, $v_i := v_{P_i}$. If $s \in \bigcap_{i=1}^{n} P_i$ (i.e. $v_i(s) \geq 1 \ \forall i$), then $[L : K(s)] \geq \sum_{i=1}^{n} v_i(s) \deg P_i$.

proof: Put $\mathcal{O}_i := \mathcal{O}_{P_i}$, $d_i := \deg P_i$, $\ell_i := v_i(s)$, $\ell := \max_{i=1..n}(\ell_i)$

By 5.20(2) $\exists\ B_i = \{b_{i1}, \dots, b_{id_i}\}$ a K-basis of $\mathcal{O}_i$ modulo P_i such that $\overset{(*)}{\boxed{v_k(b_{ij}) \geq \ell}}$ $\forall i, j \ \forall k \neq j$

By 5.19 $\forall i$ $\boxed{\exists\, t_i \in P_i : v_i(t_i) = 1 \ \& \ v_k(t_i) = 0 \ \forall k \neq i}$

Put $\widetilde{B}_i := \{ b_{ij} t_i^k \mid j = 1, \dots, d_i, k = 0, \dots, \ell_i - 1 \} \ \forall i$

$B := \bigcup_{i=1}^{n} \widetilde{B}_i$ we will show that B is LI over $K(s)$

$\Longrightarrow [L : K(s)] \geq |B| = \sum_i \ell_i d_i = \sum_i v_i(s) \deg P_i$

?? Assume that B is LD over $K(s)$ $\overset{n.t.}{\Longrightarrow}$

$\exists\, \lambda_{ijr} \in K$ $\exists\, a_{ijr} \in K[s]$ such that $\exists\, i,j,r : \lambda_{ijr} \neq 0$ &

$$\sum_i \sum_{j,r} (s \cdot a_{ijr} + \lambda_{ijr})\, b_{ij} A_i^r = 0$$

Put $c_i = \sum\limits_{\substack{j=1,\dots,d_i \\ r=0,\dots,\ell_i-1}} (s a_{ijr} + \lambda_{ijr})\, b_{ij} A_i^r \Rightarrow \sum\limits_{i=1}^{m} c_i = 0 \Rightarrow c_i = -\sum\limits_{k \neq i} c_k \Rightarrow$

$\Rightarrow v_i(c_i) = v_i\Big(-\sum\limits_{k\neq i} c_k\Big) \geq \ell \geq \ell_i$ by (*) & (DV2) $\forall i$

$\Rightarrow \sum\limits_{j,r} b_{ij} A_i^r \underbrace{(s a_{ijr} + \lambda_{ijr})}_{\in P_i^{\ell_i}} = c_i \in P_i^{\ell_i}$ & $s \in P_i^{\ell_i}$ ($\Leftarrow v(s) = \ell_i$)

$\Rightarrow \sum\limits_{j,r} \lambda_{ijr} b_{ij} A_i^r \in P_i^{\ell_i} \underset{\text{Observation (4)}}{\Longrightarrow} \forall\, i,j,r,\ \lambda_{ijr} = 0$ a contradiction

$\Rightarrow B$ is LI over $K(s)$ $\square$