AGREEMENT:

> In the rest of the section L is an AFF over K given by $f(\alpha,\beta)=0$ for $\deg f \geq 2$ which is simultaneously given by $w_\sigma(u,v)=0$ where $w_\sigma = y\,g(x,y) + h(x) + y$, $\operatorname{mult}(h) \geq 2$, $\deg(h) \geq 2$ and $\operatorname{mult} g \geq 1$, $\sigma \in \operatorname{Aff}_2(K)$, $w_\sigma = (\sigma^{-1})^*(f)$ and $(u,v) = \bar{\sigma}(\alpha,\beta)$.

! If f is smooth at $\gamma \in V_f(K)$ and $A = \begin{pmatrix} l_1 & l_2 \\ a_1 & a_2 \end{pmatrix} \in GL_2(K)$ for $a_1 = \frac{\partial f}{\partial x}(\gamma)$, $a_2 = \frac{\partial f}{\partial y}(\gamma)$, then σ could be given by $\boxed{\sigma := \theta_A \tau_\gamma}$ by the Proposition 5.7. (corrected numbering)

$\boxed{\text{T\&N}}$ Let $p \in K[x]$, $\gamma \in K$. Recall that multiplicity of (the root) γ of (the polynomial) p is $k \geq 0$ for which $(x-\gamma)^k \mid p$ and $(x-\gamma)^{k+1} \nmid p$.

Observation Let $p, s \in K[x]$, $g \in K[x,y]$, $\gamma \in K$.

(1) multiplicity of γ of p is k $\iff$ mult $\tau^*_{-\gamma}(p) = k$

(2) if $s(\gamma) = 0$, $\hat{g}(x) := g(x-\gamma, s(x))$, then multiplicity of γ of $\hat{g}(x)$ is greater or equal than mult g.

Proposition 8.9 Let $\gamma = (\gamma_1, \gamma_2) \in V_f(K)$, $\frac{\partial f}{\partial y}(\gamma) \neq 0$, $\lambda, \mu \in K$ such that $\gamma_2 = \lambda\gamma_1 + \mu$. Then $\exists!\ P \in \mathbb{P}_{L/K}$ for which $\{\alpha - \gamma_1, \beta - \gamma_2\} \subset P$ and

$v_P(\beta - \lambda\alpha - \mu)$ = multiplicity at γ of $\hat{f}(x) = f(x, \lambda x + \mu)$.

proof: Denote $\Delta = \Delta_\gamma(f) = a_1(x-\gamma_1) + a_2(y-\gamma_2)$

where $a_1 = \frac{\partial f}{\partial x}(\gamma)$, $a_2 = \frac{\partial f}{\partial y}(\gamma) \neq 0$ $\Rightarrow$

$\Rightarrow \Delta = a_2\left(y + \frac{a_1}{a_2}x - \left(\gamma_2 + \frac{a_1}{a_2}\gamma_1\right)\right)$

(+) put $\hat{\Delta}(x) := \Delta(x, \lambda x + \mu) \Rightarrow \deg_x \Delta(x) \leq 1$ &

$\hat{\Delta}(\gamma_1) = \Delta(\gamma_1, \underbrace{\lambda\gamma_1 + \mu}_{= \gamma_2 \text{ by the solutions}}) = \Delta(\gamma_1, \gamma_2) = 0 \Rightarrow$ either $(\hat{\Delta}) = (x - \gamma_1)$ or $\hat{\Delta} = 0$

(*) Note that $\hat{\Delta} = 0 \Leftrightarrow \lambda x + \mu + \frac{a_1}{a_2}x - \left(\gamma_2 + \frac{a_1}{a_2}\right) = 0 \Leftrightarrow \Delta = a_2(y - \lambda x - \mu)$

Since $a_2 \neq 0$ i.e. $\Delta \neq 0$: $\exists!\, P \in \mathbb{P}_{L|k}$: $v_P(\alpha - \gamma_1), v_P(\beta - \gamma_2) > 0$ by 5.8

$\Leftrightarrow$ $\exists!\, P : \{\alpha - \gamma_1, \beta - \gamma_2\} \subseteq P$

$A = \begin{pmatrix} 1 & 0 \\ a_1 & a_2 \end{pmatrix}$ is regular and put $\sigma = \vartheta_A \tau_{-\gamma}$ from 5.7(3) and from the proof of 5.8

$\boxed{! \; x \Leftarrow x - \gamma_1 \\ y \Leftarrow \Delta(x, y)}$

Compute $f(x, y) = \sigma^*(w_\sigma) = \sigma^*(\ell(x) + y\, g(x, y) + y) =$ ($\Delta(x,y) =$)

$= \ell(x - \gamma_1) + (a_1(x - \gamma_1) + a_2(y - \gamma_2))(g(x, \Delta(x, y)) + 1)$

substitute: $\lambda x + \mu \to y$ (in $f(x,y)$):

$$\hat f(x) := f(x, \lambda x + \mu) = h(x-\gamma_1) + \hat A(x) \cdot \left(g(x-\gamma_1, \hat A(x)) + 1\right) \quad (+)$$

(a) if $A = a_2(y - \lambda x - \mu) \Rightarrow \hat f(x) = \underbrace{h(x-\gamma_1)}_{\text{\& observation (1)}}$ by $(*)$

$\Rightarrow \underline{v_p(\beta - \lambda\alpha - \mu)} \overset{\text{by 5.8 (the proof of it)}}{=} v_p(\nu) = \text{mult } h = \underline{\text{multiplicity of } \gamma_1 \text{ of } \hat f}$

$(K(\mu,\nu) = K(\alpha,\beta))$

(b) if $A \neq a_2(y - \lambda x - \mu) \Rightarrow \hat A = c(x-\gamma_1)$ for $c \in K^*$ by $(*)$

$$\Rightarrow \hat f(x) = \underbrace{h(x-\gamma_1)}_{(x-\gamma_1)^2 \mid} + c(x-\gamma_1)\cdot \underbrace{\left(\underbrace{g(x-\gamma_1, c(x-\gamma_1))}_{(x-\gamma_1)\mid} + 1\right)}_{(x-\gamma_1) \nmid \text{ by Observ.(2)}}$$

$\Rightarrow$ multiplicity of γ_1 of $\hat f(x) = 1$

$\overset{5.8}{\Longrightarrow} \underline{v_p(\beta - \lambda\alpha - \mu)} = 1 = \underline{\text{mult of } \gamma_1 \text{ of } \hat f}$ $\square$

Example 5.10 Let $f = y^2 + xy + x^5 + 32 \in \mathbb{R}[x,y]$

$\overset{4.9}{\Longrightarrow}$ f is (absolutely) irreducible, $\alpha := x + (f) \in \mathbb{R}[x]/(f)$,
$\beta := y + (f)$ —

$\Rightarrow L = \mathbb{R}(\alpha,\beta)$ is an AFF over $\mathbb{R}$ given by $f(\alpha,\beta) = 0$

$(-2,2) \in V_f(\mathbb{R})$ as $f((-2,2)) = 4 - 4 - 32 + 32$

$\frac{\partial f}{\partial x} = y + 5x^4 \Rightarrow \frac{\partial f}{\partial (x)}(-2,2) = 82$, $\frac{\partial f}{\partial y} = 2y + x$, $\frac{\partial f}{\partial y}(-2,2) = 2$

tangent at $(-2,2)$ of f:

$\Rightarrow \boxed{t = t_{(-2,2)}(f) = 82x + 2y + 160}$, and $\boxed{\Delta = \beta + 41\alpha + 80}$

(i.e. $\Delta = \frac{1}{2}(t(\alpha,\beta))$)

$\overset{5.9}{\Longrightarrow} \exists!\, P \in \mathbb{P}_{L|K}$ s.t. $\overset{\alpha+2}{\cancel{x+2}}, \overset{\beta-2}{\cancel{y-2}} \in P$

compute $v_P(\Delta)$: $\hat{f}(x) = f(x, -41x - 80)$

$= x^5 + 40 \cdot 41 x^2 - 80 \cdot 81 x + 80^2 + 32$

$\hat{f}(-2) = 0$, $\hat{f}'(-2) = 5(-2)^4 + 80 \cdot 41(-2) + 81 \cdot 80 = 0$, $\hat{f}''(-2) \neq 0$

$\Rightarrow \boxed{v_P(\Delta) = 2}$

$\boxed{\text{T\&N}}$ Let $\underline{\gamma} \in \mathbb{A}^2(k)$ s.t. $\underline{\gamma} \in V_f(k)$

$\left(\Rightarrow (f) \subseteq I_{\underline{\gamma}} = (x-\gamma_1, y-\gamma_2) \right)$

Denote by $R_{\underline{\gamma}} := k[x,y]_{(I_{\underline{\gamma}})}$ = the localization of $k[x,y]$ at $I_{\underline{\gamma}}$

$= \left\{ \frac{a}{b} \in k(x,y) \mid a,b \in k[x,y],\ b(\underline{\gamma}) \neq 0 \right\}$

$(I_{\underline{\gamma}})$ will denote the maximal ideal of $R_{\underline{\gamma}}$

let $\omega_{\underline{\gamma}} : R_{\underline{\gamma}} \to L$ is defined by $\omega_{\underline{\gamma}}\left(\frac{a}{b}\right) = \frac{a(\alpha,\beta)}{b(\alpha,\beta)}$

thus $\frac{a}{b}(\underline{\gamma}) := \omega_{\underline{\gamma}}\left(\frac{a}{b}\right) = \frac{a(\alpha\beta)}{b(\alpha\beta)}$

Denote. ${}_f\mathcal{O}_{\underline{\gamma}} := \{ \rho \in L \mid \exists r \in R_{\underline{\gamma}} : \omega_{\underline{\gamma}}(r) = \rho \} = \mathcal{O}_{\underline{\gamma}}$

${}_fP_{\underline{\gamma}} := \{ \rho \in L \mid \exists r \in (I_{\underline{\gamma}})\ \omega_{\underline{\gamma}}(r) = \rho \} = P_{\underline{\gamma}}$ } if f is fixed

$r \in L$: $\mathrm{Dom}_f(r) = \{ \underline{\gamma} \in V_f(k) \mid r \in \mathcal{O}_{\underline{\gamma}} \}$

Observation: In the notation introduced above:

(1) ω_x is a well-defined ring homomorphism,

(2) $\mathcal{O}_x = \omega_x(R_x)$, $P_x = \omega_x((I_x))$,

(3) $\mathcal{O}_x$ is a local ring with the maximal ideal P_x,

$\mathcal{O}_x = K + P_x$ and $\dim_K \mathcal{O}_x / P_x = 1$

(4) If $\Delta = \omega_x\left(\frac{a}{\prod b_i}\right)$ where b_i are irreducible $b_i(x) \neq 0$ $\Rightarrow$

$V_f^{(k)} \setminus \mathrm{Dom}_f(\alpha) = \{\alpha \in V_f^{(k)} \mid \Delta \notin \mathcal{O}_\alpha\} \subseteq \bigcup V_{\{f, b_i\}}$ - finite by 4.4.

$\nwarrow$ finite

(5) ${}_f\mathcal{O}_x = {}_{\tau^*_x(f)}\mathcal{O}_{(0,0)}$ and ${}_f P_x = {}_{\tau^*_x(f)} P_{(0,0)}$