Observation: $\forall \sigma \in Aff_2(K)\ \exists!\ \overline{\sigma} \in Aff_2(L)$ $\boxed{K \subseteq L}$

such that $\overline{\sigma}(\gamma) = \sigma(\gamma)\ \forall \gamma \in A^2(K)$.

$\boxed{T\&N}$ In the sequel $\overline{\sigma}$ will denote the extension of $\sigma \in Aff_2(K)$ to $Aff_2(L)$ from the observation.

Observation (A) Let $a, f \in K[x,y] \subseteq L[x,y]$ and $\sigma \in Aff_2(K)$

(1) $(\sigma^{-1})^*(a(\sigma^*(x), \sigma^*(y))) = a(\sigma^*((\sigma^{-1})^*(x)), \sigma^*((\sigma^{-1})^*(y))) = a(x,y)$,

(2) if $f(x,y) = a(\sigma^*(x), \sigma^*(y)) \overset{(1)}{\Longrightarrow} a = (\sigma^{-1})^*(f)$,

(3) $(\overline{\sigma})^*(a) = \sigma^*(a) \in K[x,y]$.

Observation (B): Let $\sigma \in Aff_2(K)$, $\tilde{\alpha}, \tilde{\beta} \in L$, $\mu := \overline{\sigma}^*(x)(\tilde{\alpha}, \tilde{\beta})$, $\nu := \overline{\sigma}^*(y)(\tilde{\alpha}, \tilde{\beta})$. Then

(1) $(\mu, \nu) \overset{A(3)}{=} (\overline{\sigma}^*(x)(\tilde{\alpha}, \tilde{\beta}), \overline{\sigma}^*(y)(\tilde{\alpha}, \tilde{\beta})) = \overline{\sigma}(\tilde{\alpha}, \tilde{\beta})$,

(2) $(\tilde{\alpha}, \tilde{\beta}) \overset{(1)}{=} \overline{\sigma}^{-1}(\mu, \nu) \Longrightarrow K(\tilde{\alpha}, \tilde{\beta}) = K(\mu, \nu)$,

(3) $(\sigma^{-1})^*(f)(\mu, \nu) = f(\overline{\sigma}^{-1}(\mu, \nu)) \overset{(1)}{=} f(\tilde{\alpha}, \tilde{\beta})$.

Lemma 5 ~~7~~ (correction!) Let L be an AFF over k given by $f(\alpha,\beta)=0$, $\gamma=(\gamma_1,\gamma_2)\in\mathbb{A}_2(k)$, $A\in GL_2(k)$, $\sigma=\vartheta_A\tau_{-\gamma}$, $(u,s)=\bar{\sigma}(\alpha,\beta)$, $w_\sigma:=(\sigma^{-1})^*(f)$. Then

(1) L is an AFF over k given by $w_\sigma(u,s)=0$.

(2) If f is smooth at $\gamma\in V_f(k)$, then $\exists A$ such that either $w_\sigma=y$

or $w_\sigma=h(x)+y\,g(x,y)+y$ where $h\in k[x]$, $g\in k[x,y]$, $h\neq 0$, $\mathrm{mult}(h)\geq 2$, $\mathrm{mult}\,g\geq 1$.

(3) Let $s_\gamma(f)=a_1(x-\gamma_1)+a_2(y-\gamma_2)$ for $\gamma\in V_f(k)$.

Then A is a matrix from (7) (i.e. if $\sigma=\vartheta_A\tau_\gamma$ determines the polynomial w_σ from (7)) if and only if $A=\begin{pmatrix} b_1 & b_2\\ \boxed{a_1 & a_2}\end{pmatrix}$.

Proof: (1) $(\sigma^{-1})^*$ is isomorphism & f is irreducible $\Rightarrow$

$\Rightarrow (\sigma^{-1})^*(f)$ is irreducible

by Obs. B(3): $w_\sigma(u,v) = f(\alpha,\beta) = 0$

by Obs. B(2): $K(u,v) = K(\alpha,\beta) = L$

$\Rightarrow L$ is given by $w_\sigma(u,v) = 0$

(2) (i) $\gamma \in V_f(K) \Rightarrow \exists a_1, a_2 \in K \; \exists \tilde{f} \in K[x,y]$ with mult $\tilde{f} \geq 2$

such that $f(x,y) = \underbrace{a_1(x-\gamma_1) + a_2(y-\gamma_2)} + \tilde{f}(x-\gamma_1, y-\gamma_2)$

(ii) by 3.8: $\Delta_\gamma(f) = \tau_\gamma(L(\tau_\gamma(f))) \overset{\Downarrow}{=} a_1(x-\gamma_1) + a_2(y-\gamma_2)$

(iii) f is smooth at $\gamma \Rightarrow \Delta_\gamma(f) \neq 0 \Rightarrow (a_1, a_2) \neq (0,0)$

Let $A = \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \in GL_2(K) \Rightarrow \Delta := \det A \neq 0 \Rightarrow$

$\Rightarrow A^{-1} = \Delta^{-1} \cdot \begin{pmatrix} c_2 & -b_2 \\ -c_1 & b_1 \end{pmatrix}$

Since $\sigma = \vartheta_A \tau_{-\gamma} \Rightarrow \sigma^{-1} = \tau_\gamma \vartheta_{A^{-1}} \Rightarrow$

$\Rightarrow (\sigma^{-1})^* = (\vartheta_{A^{-1}})^* (\tau_\gamma)^*$

Compute: $(\sigma^{-1})^*(\ell) = (\vartheta_{A^{-1}})^*(\tau_\ell)^*(\ell) \overset{(i)}{=} (\vartheta_{A^{-1}})^*(a_1x + a_2y + \tilde{f}) =$

$= \Delta^{-1}(a_1(c_2x - b_2y) + a_2(-c_1x + b_1y)) + \underbrace{(\vartheta_{A^{-1}})^*(\tilde{f}(x,y))}_{\hat{f} :=\ \boxed{\text{mult } \hat{f} \geq 2}} =$

$= \Delta^{-1}(\underbrace{(a_1c_2 - a_2c_1)}_{=0 \iff \exists d : (a_1, a_2) = d(c_1, c_2)}x + \underbrace{(b_1a_2 - b_2a_1)}_{= \Delta \iff d = 1}y) + \hat{f} =$

$= y + \hat{f} \overset{(+)}{\iff} A = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix} \in GL_2(K)$

As $\text{mult } \hat{f} \geq 2$ $\exists\, h \in K[x],\ g \in K[x,y] : \hat{f} = h(x) + y\, g(x,y)$

& $\text{mult } h \geq 2$ $\underline{\text{mult } g \geq 1}$

If $h = 0 \Rightarrow w_\sigma = \underline{(\sigma^{-1})^*(\ell)} = y \cdot (1 + g(x,y)) \Rightarrow y$

irreducible $\Rightarrow g = 0$

(3) follows by (+).

<u>Theorem S8</u> Let L be an AFF over k given by $f(\alpha,\beta)=0$ and f be smooth at $\gamma=(\gamma_1,\gamma_2)\in V_f(k)$.

(1) Then $\exists!\, P\in \mathbb{P}_{L|k}$ such that $v_P(\alpha-\gamma_1)>0$, $v_P(\beta-\gamma_2)>0$.

(2) If $l(x,y)=l_0+l_1x+l_2y\in k[x,y]$, then for P from (1) holds:

$$v_P(l(\alpha,\beta))\begin{cases} =0 & \text{if } l(\gamma)\neq 0,\\ =1 & \text{if } l(\gamma)=0 \ \& \ l\notin(\Delta_\gamma(f)),\\ \geq 2 & \text{if } \text{— " —} \ \& \ l\in(\Delta_\gamma(f)).\end{cases}$$

<u>proof</u>: Let $\Delta_\gamma(f)=a_1(x-\gamma_1)+a_2(y-\gamma_2)\neq 0$ since f is smooth at γ

$\xRightarrow{\text{LA}} \exists\,(b_1,b_2): A=\begin{pmatrix} b_1 & b_2\\ a_1 & a_2\end{pmatrix}\in GL_2(k)$, put $\sigma:=\vartheta_A\,\tau_{-\gamma}$ & $(\mu,\nu):=\sigma(\alpha,\beta)$

$\xRightarrow{\text{S.7}} h_\sigma=(\sigma^{-1})^*(f)=$ either y or $h(x)+yg(x,y)+y$ s.t. the problem of S.5 holds.

if $v_\sigma \neq v_\gamma \overset{5.7}{\Longrightarrow} \exists!\, P \in \mathbb{P}_{L/K} : v_P(\mu) > 0,\ v_P(\Delta) > 0$
$(\ \&\ v_P(\Delta) > v_P(\mu)\ )$

if $v_\sigma = v_\gamma \Rightarrow \beta = 0 \Rightarrow L = K(\alpha) \cong K(x)$

$\Rightarrow \exists!\, P \supseteq (\alpha) : v_P(\alpha) > 0 \quad (P = (\alpha) \cap \mathcal{O}_P)$

$\Rightarrow v_P(\alpha) = 1 \ \&\ v_P(\beta) = v_P(0) = \infty > v_P(\alpha)$

Observe that $\begin{pmatrix} \mu \\ \Delta \end{pmatrix} = A \begin{pmatrix} \alpha - \gamma_1 \\ \beta - \gamma_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha - \gamma_1 \\ \beta - \gamma_2 \end{pmatrix} = A^{-1} \begin{pmatrix} \mu \\ \Delta \end{pmatrix}$

$\Rightarrow \mathrm{Span}_K(\mu, \Delta) = \mathrm{Span}_K(\alpha - \gamma_1, \beta - \gamma_2) \Rightarrow$

$\Rightarrow \ell(\alpha, \beta) - \underbrace{\ell(\gamma_1, \gamma_2)}_{\in K} = \ell_1(\alpha - \gamma_1) + \ell_2(\beta - \gamma_2) \in \mathrm{Span}_K(\mu, \Delta) \subseteq P$

Hence $\boxed{v_P(\ell(\alpha, \beta)) > 0} \Leftrightarrow \ell(\alpha, \beta) \in P \Leftrightarrow \ell(\gamma_1, \gamma_2) \in P\ (\cap K)$

$\underset{P \cap K}{\Longleftrightarrow} \boxed{\ell(\gamma_1, \gamma_2) = 0}$

We have proved
$v_P(\ell(\alpha, \beta)) = 0$ if $\ell(\gamma) \neq 0$

Let $l(\gamma)=0$ in the rest of the proof $\Rightarrow$

$$\Rightarrow l(\alpha,\beta) = l_1(\alpha-\gamma_1) + l_2(\beta-\gamma_2) \in \mathrm{Span}_K(\mu,\Delta)$$

$$\left.\begin{array}{l} v_p(\mu)=1 \\ v_p(\Delta)\geq 1 \end{array}\right\} \overset{2.13}{\Longrightarrow} v_p(l(\alpha,\beta)) > 1 \Leftrightarrow l_1(\alpha-\gamma_1) + l_2(\beta-\gamma_2) \in \mathrm{Span}(\Delta)$$

$$\Leftrightarrow (l_1, l_2)\, A^{-1}\begin{pmatrix}\mu\\ \Delta\end{pmatrix} = (l_1, l_2)\begin{pmatrix}\alpha-\gamma_1\\ \beta-\gamma_2\end{pmatrix} \in \mathrm{Span}(\Delta)$$

$$\Leftrightarrow \exists c\in K : (l_1, l_2)\underset{\Delta^{-1}\begin{pmatrix} a_2 & -b_2 \\ -a_1 & b_1\end{pmatrix}}{A^{-1}} = (0, c)$$

$$\Leftrightarrow (l_1\ l_2)\begin{pmatrix}a_2\\ -a_1\end{pmatrix} = 0$$

$$\Leftrightarrow \exists d\in K : (l_1, l_2) = d(a_1, a_2)$$

$$\Leftrightarrow l \in (\Delta_{\gamma}(\beta))$$