During the whole lecture:

$\boxed{\begin{array}{l} K \text{ is a field} \\ \overline{K} \text{ is an algebraic closure of } K \\ \mathbb{X} = \{x_1, \dots, x_n\} \text{ (a set of variables)} \end{array}}$

$\boxed{\text{T\&N}}$ Let $K \subseteq L \subseteq \overline{K}$ be field extensions

Denote: $\mathbb{A}^n := \overline{K}^n$ (the affine space = a space of points)

$\mathbb{A}^n(L) := L^n$ – <u>L-rational points of $\mathbb{A}^n$</u>

Let $M \subset K[\mathbb{X}]$ $\quad V_M := \{\underline{\alpha} \in \mathbb{A}^n \mid a(\underline{\alpha}) = 0 \ \forall a \in M\}$ – „common points of M"

$V_M(L) := V_M \cap \mathbb{A}^n(L)$ – <u>L-rational points of M</u>

$V_a := V_{\{a\}}$, $V_a(L) := V_{\{a\}}(L)$

<u>Observation</u>: Let $M \subset K[\mathbb{X}]$, $a \in K[\mathbb{X}]$, $\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{A}^n$

(1) $V_M = V_{(M)}$

(2) $\operatorname{mult} \tau_{\underline{\beta}}^*(a) \geq 1 \iff \operatorname{mult}(a(x+\beta_1, \dots, x+\beta_n)) \geq 1 \iff$

$\iff a(\beta_1, \dots, \beta_n) = 0 \iff \underline{\beta} \in V_a$

$\boxed{\text{T\&N}}$ Let $a = \sum_{i_1 \dots i_n} a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} = \underbrace{\sum_{j \geq 0} b_j x_i^j}_{\text{fix } i \in \{1, \dots, n\}} \in K[\underline{x}]$

where $b_j \in K[\underline{x} \setminus \{x_i\}]$

Define: $L(a) := \sum_{i_1 \dots i_n : \sum i_j = 1} a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} = \sum_{j=1}^{n} a_{\delta_{1j} \dots \delta_{nj}} x_j$ - the linear part of a

$\frac{\partial a}{\partial x_i} := \sum_{j \geq 0} (j+1) b_{j+1} x_i^j$ - (partial) derivative (in variable x_i)

Let $\underline{\alpha} \in V_a$ (i.e. $a(\underline{\alpha}) = 0$) and put $c_i := \frac{\partial a}{\partial x_i}(\underline{\alpha})$ $(\in \overline{K})$

We say that a is smooth at $\underline{\alpha}$ if $\exists i : c_i \neq 0$

a is singular at $\underline{\alpha}$ if $\forall i \; c_i = 0$

(tangent!)
$t_{\underline{\alpha}}(a) := \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{n} c_i \alpha_i = \sum_{i=1}^{n} c_i (x_i - \alpha_i)$ $(\in \overline{K}[\underline{x}])$

Observation Let $a \in K[\underline{X}]$, $\underline{\alpha} \in \mathbb{A}^n$

(1) a is smooth at $\underline{\alpha}$ $\Leftrightarrow$ $\Delta_{\underline{\alpha}}(a) \neq 0$

(2) $\underline{\alpha} \in V_{\Delta_{\underline{\alpha}}(a)}$

Example 3.7 Let $w = y^2 - (x^3 + x - 2)$ (be a short WBP) $\in \mathbb{R}[x, y]$

Then: $L(w) = -x$, $\dfrac{\partial w}{\partial x} = -3x^2 - 1$, $\dfrac{\partial w}{\partial y} = 2y$

$w\left(\begin{pmatrix}1\\0\end{pmatrix}^T\right) = 0 - 0 = 0 \Rightarrow (1,0) \in V_w$

for $\underline{\alpha} = (1,0)$: $c_1 = -4$, $c_2 = 0 \Rightarrow \Delta_{(1,0)}(w) = -4(x-1)$

Lemma 3.8 Let $a \in K[\underline{X}]$, $\underline{\alpha} \in V_a$. Then

$$\Delta_{\underline{\alpha}}(a) = \tau^*_{-\underline{\alpha}}(L(\tau^*_{\underline{\alpha}}(a)))$$

Proof: Put $c := \sum c_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} := \tau^*_{\underline{\alpha}}(a) \in \overline{K}[\underline{X}]$

Then $a = \tau^*_{-\underline{\alpha}}(c) = c(x_1-\alpha_1, \dots, x_n-\alpha_n)$

Since $\frac{\partial a}{\partial x_i}(\underline{\alpha}) = c_{\delta_{1i}\cdots\delta_{ni}}$ & $\sum c_{\delta_{1i}\cdots\delta_{ni}} x_i = L(c) = L(\tau^*_{\underline{\alpha}}(a))$

use the substitution $x_i \leftarrow x_i - \alpha_i$:

$A_{\underline{\alpha}}(a) = \sum c_{\delta_{i1}\cdots\delta_{in}}(x_i-\alpha_i) = \tau^*_{-\underline{\alpha}}(L(c)) = \tau^*_{-\underline{\alpha}}(L(\tau^*_{\underline{\alpha}}(a)))$

$\boxed{\text{T\&N}}$ Let $a \in k[x,y]$, $\deg a \geq 1$ (recall: $\deg \sum a_{v_1\dots v_n} x_1^{v_1}\cdots x_n^{v_n} = \max\{\sum v_i \mid a_{v_1\dots v_n} \neq 0\}$)

$C = V_a$ – an affine plane curve

$\underline{\alpha} \in C$ is smooth/singular point of C if a is smooth/singular at $\underline{\alpha}$.
(a singularity of C = a singular point of C)

a is smooth if a is smooth at $\underline{\alpha}$ $\forall \underline{\alpha} \in V_a$

a is singular if $\exists \underline{\alpha} \in V_a$: a is singular at $\underline{\alpha}$

$C (= V_a)$ is smooth / singular if a is smooth / singular

Lemma 3.9 Let $a \in \overline{K}[\underline{X}]$, $\underline{\alpha} \in \mathbb{A}^n$, $\sigma \in \mathrm{Aff}_n(\overline{K})$

Then $\Delta_{\underline{\alpha}}(\sigma^*(a)) = \sigma^*(\Delta_{\sigma(\underline{\alpha})}(a))$

proof: we know that $\exists \underline{\gamma} \in \mathbb{A}^n$, $A \in \overline{K}^{n\times n}$ regular: $\sigma = \tau_{\underline{\gamma}} \vartheta_A$

$$\Rightarrow \sigma\tau_{\underline{\alpha}} = \tau_{\underline{\gamma}}\vartheta_A\tau_{\underline{\alpha}} \overset{\text{Observation}}{=} \underbrace{\tau_{\underline{\gamma}}\tau_{\vartheta_A(\underline{\alpha})}}_{=\tau_{\underline{\gamma}+\vartheta_A(\underline{\alpha})}}\vartheta_A = \tau_{\sigma(\underline{\alpha})}\vartheta_A \qquad (*)$$

$$\&: \tau_{-\sigma(\underline{\alpha})}\sigma = \underbrace{\tau_{-\underline{\gamma}-A(\underline{\alpha})}\tau_{\underline{\gamma}}}_{=\tau_{-\underline{\gamma}-A(\underline{\alpha})+\underline{\gamma}}}\vartheta_A = \tau_{A(-\underline{\alpha})}\vartheta_A = \vartheta_A\tau_{-\underline{\alpha}} \qquad (**)$$

$$\text{by 3.8}: \Delta_{\underline{\alpha}}(\sigma^*(a)) = \tau_{-\alpha}^* L(\overbrace{\tau_\alpha^*(\sigma^*(a))}^{(\sigma\tau_\alpha)^*(a)}) \overset{(*)}{=} \tau_{-\alpha}^* L(\vartheta_A^*\tau_{\sigma(\underline{\alpha})}^*(a)) =$$

$$= \underbrace{\tau_{-\underline{\alpha}}^*\vartheta_A^*}_{(\vartheta_A\tau_{-\underline{\alpha}})^*} L(\tau_{\sigma(\underline{\alpha})}^*(a)) \overset{(**)}{=} \sigma^*\tau_{-\sigma(\underline{\alpha})}^* L(\tau_{\sigma(\underline{\alpha})}^*(a))$$

! it commutes as ϑ_A is linear!

$$\overset{3.8}{=} \sigma^*(\Delta_{\sigma(\alpha)}(a))$$

on $\Delta_{\sigma(\alpha)}(a)$

[T&N] If $\sigma = \tau_{\underline{b}}\vartheta_A \in Aff_n(K)$, then denote by $\boxed{\bar\sigma \in Aff_n(\bar K)}$ corresponding aff. mapping $\tau_{\underline{b}}\vartheta_A$ on $\bar K^n$
($\underline{b} \in K^n \subseteq \bar K^n$, $A \in GL_n(K) \subseteq GL_n(\bar K)$)

Lemma 3.10 Let $a \in K[\underline{X}]$, $\sigma \in Aff_n(K)$

(1) $\bar\sigma(V_{\sigma^*(a)}) = V_a$

(2) $\sigma^*(a)$ is singular at $\underline{\alpha} \in V_{\sigma^*(a)} \iff$

$\iff$ a is singular at $\bar\sigma(\underline{\alpha})$

proof: (1) $\underline{\alpha} \in V_{\sigma^*(a)} \overset{\text{by definition}}{\iff} \sigma^*(a)(\underline{\alpha}) = 0 \iff a(\bar\sigma(\underline{\alpha})) = 0 \iff$
$\iff \bar\sigma(\alpha) \in V_a$

(2) follows by (the proof of) (1) and 3.9 since

$$\sigma^*(\Delta_\beta(a)) = 0 \iff \Delta_\beta(b) = 0$$

Corollary 3.11 Let w and $\tilde w$ ~~be~~ be K-equivalent WEP's

Then w is smooth ~~nonsingular~~ $\iff$ $\tilde w$ is smooth

proof: by 3.10(2) & 3.4

$\boxed{T\&N}$ Weierstrass equation $w=0$ is smooth/singular

if WEP w is smooth/singular.

Recall: $f \in K[x]$ is separable $\overset{def}{\Leftrightarrow}$ f has no multiple roots (in $\bar{K}$), K is perfect $\overset{def}{\Leftrightarrow}$ irreducible polynomials are separable

<u>Proposition 3.12</u> Let char $K \neq 2$ and $w = y^2 - f(x)$ be a short WEP $\in K[x,y]$ $(f(x) \in K[x])$

(1) w has at most 1 singularity

(2) if K is perfect, then any singularity is K-rational, (i.e. $\in K^2$)

(3) w is smooth $\Leftrightarrow$ f is separable

<u>proof</u>: Note that $\frac{\partial w}{\partial x} = f'(x)$ and $\frac{\partial w}{\partial y} = 2y$

(1) Let w be singular at $\alpha = (\alpha_1, \alpha_2) \in V_w$ $\Rightarrow$

$\Rightarrow$ $w(\underline{\alpha}) = 0$ & $f'(\alpha_1) = 0$ & $\overset{\neq 0}{2}\alpha_2 = 0$ (3 Nullpunkte) $\overset{\text{char} K \neq 2}{\Longrightarrow}$ $\alpha_2 = 0$

$\overset{w((\alpha_1, 0)) = 0}{\Longrightarrow}$ $f(\alpha_1) = 0 = f'(\alpha_1)$ $\Rightarrow$ $(x-\alpha_1)^2 / f$ $\Rightarrow$ f is not separable

Since $\deg f = 3$ & α_1 is a root of multiplicity 2 $\Rightarrow \exists!$ such α_1

$\Rightarrow$ if singularity $\underline{\alpha}$ exists then it is uniquely defined

(2) Let K be perfect and $f = (x-\alpha_1)^2(x-\beta)$

(a) $\alpha_1 \neq \beta \Rightarrow (x-\alpha_1) = \mathrm{GCD}(f, f') \in K[x] \Rightarrow \alpha_1 \in K$

(b) $\alpha_1 = \beta \Rightarrow f = (x-\alpha_1)^3$ is reducible $\Rightarrow (x-\alpha_1) \in K[x] \Rightarrow \alpha_1 \in K$

$\Rightarrow \underline{\alpha} = (\alpha_1, 0) \in K^2$ (if $\underline{\alpha}$ is a singularity)

(3) ($\Rightarrow$) f is not smooth $\Rightarrow$ f is not separable is done in (1)

($\Leftarrow$) if f is not separable $\Rightarrow \exists \alpha_1 \in \overline{K} : (x-\alpha_1)^2 / f$

$\Rightarrow f(\alpha_1) = 0 = f'(\alpha_1) \underset{\text{(as in (1))}}{\Rightarrow} (\alpha_1, 0)$ is a singularity of w

$\Rightarrow$ f is not smooth

<u>Example 3.13</u> (1) $y^2 - \underbrace{(x^3+1)}_{\text{separable}} \in \mathbb{R}[x,y]$ is smooth WEP

(2) $y^2 - (x^3 - x^2 - x + 1) = y^2 - (x-1)^2(x+1)$ is singular (singularity : $(1,0)$).