In the sequel L is an AFF over K given by $f(\alpha,\beta)=0$ for α,β transcendental | 14.4

Comment: The case β - algebraic means that $K(\alpha\beta)\cong\tilde{K}(\alpha)$ which is trivial from the point of view places and we have described in 2.14

Observation

Let $\omega: K[x,y]\to K[\alpha\beta]$ be substitution mapping $\omega(m):=m(\alpha,\beta)$, and $P\neq 0$ be a prime ideal of $K[\alpha\beta]$

(1) ω is a surjective ring homomorphism and $\operatorname{Ker}\omega=(f)$.

(2) $(f)\subsetneq\omega^{-1}(P)$ and $\omega^{-1}(P)$ is a prime ideal of $K[x,y]$ $\overset{4.3}{\Rightarrow}$ $\Rightarrow \exists \underline{x}\in V_f$ such that $P=\omega(I_{\underline{x}})$

(3) $\hat{K}:=K[\alpha\beta]/P=\hat{K}[\alpha+P,\beta+P]$ is K-algebra, P is maximal $\Rightarrow$ $\hat{K}$ is an algebraic extension of K $\Rightarrow [\hat{K}:K]<\infty$.

(4) $[\hat{K}:K]=1 \underset{(2)}{\overset{1.2}{\Leftrightarrow}} \underset{K[\alpha\beta]/P}{\overset{\cong}{K[x,y]/I_{\underline{x}}}}\cong K \overset{4.3}{\Leftrightarrow} \underline{x}\in V_f(K)$

Comment: We describes K-rational $\underline{x}$ using maximal ideals P

Lemma 5.15 Let $P\in\mathbb{P}_{L|K}$, $\tilde{P}:=P\cap K[\alpha\beta]$

(1) $K[\alpha,\beta]\subseteq\mathcal{O}_P \Rightarrow \tilde{P}$ is maximal in $K[\alpha\beta]$, $\dim_K K[\alpha\beta]/\tilde{P}<\infty$ and $v_P(\alpha)\geq 0$, $v_P(\beta)\geq 0$

(2) $K[\alpha\beta]\not\subseteq\mathcal{O}_P \Rightarrow \tilde{P}=0$ and either $v_P(\alpha)<0$ or $v_P(\beta)<0$

(3) — " — & f is a WEP $\Rightarrow v_P(\alpha)<0$ & $v_P(\beta)<0$ and $3v_P(\alpha)=2v_P(\beta)$

Comment: (1) and (2) characterises the correspondance between primes of $K[\alpha\beta]$ and places

Proof (1) $\alpha\beta\in\mathcal{O}_P \Rightarrow v_P(\alpha)\geq 0$, $v_P(\beta)\geq 0$, by Observation (3) it is enough to show $\tilde{P}\neq 0$.

?? $\tilde{P}=0 \Rightarrow K[\alpha\beta]\setminus\{0\}\subseteq\mathcal{O}_P\setminus P=\mathcal{O}_P^*$

$\Rightarrow$ all elements from $K[\alpha\beta]\setminus\{0\}$ are invertible in $\mathcal{O}_P \Rightarrow K(\alpha\beta)\subseteq\mathcal{O}_P$ - a contradiction

Now we can apply Observ. (3) as $\tilde{P}\neq 0$

(2) proving by contradiction ?? ~~Let~~ assume that $\tilde{P}\neq 0$

$\exists a\in K[\alpha,\beta]\setminus\mathcal{O}_P \Rightarrow v_P(a)<0$, by Observation (3) $a+\tilde{P}$ is algebraic over K $\Rightarrow$

$\Rightarrow \exists m\in K[x]$ of $\deg m\geq 1$: $m(a)\in\tilde{P}\subseteq P \Rightarrow v_P(m(a))\geq 1$

but $v_P(a)<0 \overset{2.17}{\Rightarrow} v_P(m(a))=\deg m\cdot v_P(a)<0$ ← a contradiction

$\Rightarrow \tilde{P}=0$; indirectly: $v_P(\alpha)\geq 0, v_P(\beta)\geq 0 \Rightarrow \alpha\beta\in\mathcal{O}_P \Rightarrow K[\alpha\beta]\subseteq\mathcal{O}_P$

(3) Let $f=y^2+yg(x)+h(x)$ for $g,h\in K[x]$, $\deg g\leq 1$, $\deg h=3$

Since $f(\alpha,\beta)=0 \Rightarrow \beta(\beta+g(\alpha)) = h(\alpha)$; Put: $\boxed{a := h(\alpha),\ b := \beta(\beta+g(\alpha))}$, $v := v_P$

Clearly: $v(a) = v(b) \overset{DV1}{=} v(\beta) + v(\beta+g(\alpha))$

We prove that $v(\alpha)<0 \Rightarrow v(\beta)<0$ and $v(\beta)<0 \Rightarrow v(\alpha)<0$

We use computing rules of DV: (DV1)-(DV3), 2.13, 2.17

(a) Assume the contrary ?? $v(\alpha) < 0 \le v(\beta)$

$\Rightarrow \underbrace{3v(\alpha)}_{\deg h} \overset{2.17}{=} v(a) = v(b) = \underbrace{v(\beta)}_{>v(\alpha)} + \underbrace{v(\beta+g(\alpha))}_{>v(\alpha)} > 2v(\alpha) \Rightarrow v(\alpha) > 0 \Rightarrow$ a contradiction

(b) Assume ?? $v(\alpha) \ge 0 > v(\beta) \Rightarrow \boxed{0 \le v(a) \overset{2.17}{=} v(b) = \overbrace{v(\beta)}^{<0} + v(\beta+g(\alpha)) \overset{2.13}{=} 2v(\beta) < 0}$

Hence (a)&(b) $\Rightarrow \underline{v(\alpha)<0\ \&\ v(\beta)<0}$

(c) Assume ?? $v(\alpha) \le v(\beta) \Rightarrow 3v(\alpha) = v(a) = v(b) = \overbrace{v(\beta)}^{\ge v(\alpha)} + \overbrace{v(\beta+g(\alpha))}^{\ge v(\alpha)} \ge 2v(\alpha) \Rightarrow$ $\Rightarrow$ a contradiction

$\Rightarrow v(\alpha) > v(\beta) \Rightarrow \underline{3v(\alpha)} = v(a) = v(b) = v(\beta) + v(\underbrace{\beta+g(\alpha)}_{v(\beta)<v(g(\alpha))}) = 2v(\beta)$

$\Rightarrow v(\alpha) \ge 0 \Rightarrow$ a contradiction

Proposition 5.16 Let $P \in \mathbb{P}_{L|k}$, $\deg P = 1$, f be smooth at $V_f(k)$. (TFAE)

(1) $k[\alpha,\beta] \subseteq \mathcal{O}_P$,

(2) $\exists!\ \gamma = (\gamma_1,\gamma_2) \in V_f(k)$ such that $v_P(\alpha-\gamma_1) > 0$, $v_P(\beta-\gamma_2) > 0$,

(3) $\exists!\ \gamma \in V_f(k)$ such that $P = P_\gamma$.

Comment: 5.16 describes all places of deg 1 with $v_P(\alpha), v_P(\beta) \ge 0$ as ideals P_γ for $\gamma \in V_f(k)$, but $\exists$ also places of deg >1!!

Proof: (1)$\Rightarrow$(2) $\exists$ Put $\tilde{P} := P \cap k[\alpha,\beta] \overset{5.15(1)}{\Longrightarrow} \tilde{P}$ is maximal $= \omega(A_\gamma)$ for some $\gamma \in V_f$ Obser. (2)

Note that $0 \neq k[\alpha,\beta]/\tilde{P} \underset{\text{3rd Isom. Thm}}{\cong} (k[\alpha,\beta]+P)/P$ is a subspace of k-space $\mathcal{O}_P/P \Rightarrow$

$\Rightarrow 0 < \dim_k(k[\alpha,\beta]/\tilde{P}) \le \dim \mathcal{O}_P/P = \deg P = 1 \Rightarrow \dim_k(k[\alpha,\beta]/\tilde{P}) = 1$

$\overset{\text{Observ.(4)}}{\Longrightarrow} \gamma \in V_f(k) \Rightarrow$ the existence is proved. Unicity: $\left.\begin{matrix}\alpha-\gamma_1, \beta-\gamma_2 \in P\\ \alpha-\tilde\gamma_1, \beta-\tilde\gamma_2 \in P\end{matrix}\right\} \Rightarrow \begin{matrix}\gamma_1-\tilde\gamma_1 \in P\\ \gamma_2-\tilde\gamma_2 \in P\end{matrix}$

(2)$\Rightarrow$(3) by 5.13 (unicity clear!)

(3)$\Rightarrow$(1) since $\alpha-\gamma_1, \beta-\gamma_2 \in P_\gamma = P$ and $\gamma_1,\gamma_2 \in k \Rightarrow \alpha,\beta \in k + P_\gamma = \mathcal{O}_\gamma \subseteq \mathcal{O}_P$

$\Rightarrow k[\alpha,\beta] \subseteq \mathcal{O}_P$

Corollary 5.17 If f a WEP is smooth at γ $\forall \gamma \in V_f(k)$ and $P \in \mathbb{P}_{L|k}$, $\deg P = 1$, then either $\exists \gamma \in V_f(k)$: $P = P_\gamma$ or $\alpha^{-1}, \beta^{-1} \in P$.

Proof: It follows from 5.16 & 5.15(3).

Comment: Later we will show that for f WEP the condition $\alpha^{-1},\beta^{-1} \in P$ $\Rightarrow \deg P = 1$ & $\exists!$ such P which finishes the description of places of $\deg = 1$

Observation: Let $\tilde{k}$ be the field of constants.

Recall: $\tilde{k} = \{a \in L \mid a$ - algebraic over $k\}$ $\Rightarrow L - \tilde{k}$ = all transcendent elements of L

(1) If $s \in L - \tilde{k} \overset{2.5}{\Rightarrow} \exists P \in \mathbb{P}_{L|k}$: $v_P(s) > 0$

(2) $\tilde{k} = \{a \in L \mid v_P(a) = 0\ \forall P \in \mathbb{P}_{L|k}\}$ (1) $\Rightarrow$ "$\supseteq$" and 2.15(1) $\Rightarrow$ "$\subseteq$"

(3) If $a, b \in L$, $P \in \mathbb{P}_{L|k}$, $v_P(a) \neq 0 \neq v_P(b) \overset{2.13\ \&(DV1)}{\Longrightarrow}$

$\Rightarrow v_P(a+b^k) = \min(v_P(a), k v_P(b))$ for all k except at most one k

$\Rightarrow \exists k_0: \forall k \ge k_0$ the equality holds true.

Lemma 5.18 Let $P_1, \dots, P_n \in \mathbb{P}_{L|K}$ be pairwise distinct, $n \geq 1$, $v_i := v_{P_i}$ $\forall i = 1, \dots, n$, $a_1, \dots, a_n \in L$, $r \in \mathbb{Z}$. Then:

(1) $\exists s \in L^*$: $v_1(s) > 0$ and $v_i(s) < 0$ $\forall i = 2, \dots, n$

(2) $\exists t_r \in L$: $v_i(t_r - a_i) > r$ $\forall i = 1, \dots, n$

Comment: This technical lemma (with quite "ugly" proof based on lovely computation) which is needed for important Weak Approximation Theorem (WAT). The formulation contains two steps of proof; we will use the result (2) later. We will proof WAT first and then we will prove this lemma.

Theorem 5.19 (Weak Approximation Theorem)

Let $n \geq 1$ and $P_1, \dots, P_n \in \mathbb{P}_{L|K}$ be pairwise distinct. If $a_1, \dots, a_n \in L$ and $r_1, \dots, r_n \in \mathbb{Z}$, then $\exists s \in L$ such that $v_{P_i}(s - a_i) = r_i$ $\forall i = 1, \dots, n$

Comment: 5.19 says that we can find for any finite system of places $P_1, \dots, P_n$ and elements $a_1, \dots, a_n$ a single element s for which $s \equiv a_i \pmod{P_i^{r_i}}$ where "the depth" r_i (we have $P_i \supsetneq P_i^2 \supsetneq P_i^3 \supsetneq \dots \supsetneq P_i^{r_i}$) is arbitrary! (That is the approximation)

Proof: Fix $b_i \in L$ such that $v_i(b_i) = r_i$ $\forall i = 1, \dots, n$ (we can do it since $P_i^{r_i} \setminus P_i^{r_i+1} \neq 0$ and $b_i \in P_i^{r_i} \setminus P_i^{r_i+1}$)

We use 5.18(2) two times: [put $v_i := v_{P_i}$, $r := \max\{r_i \mid i = 1, \dots, n\}$]

$\exists t \in L$: $v_i(t - b_i) > r \geq r_i$ $\forall i = 1, \dots, n$ (by 5.18(2)) and

$\exists s \in L$: $v_i(s - (t + a_i)) > r \geq r_i$ $\forall i = 1, \dots, n$ — " —

$$\Rightarrow s - a_i = \underbrace{(s - (t + a_i))}_{> r_i} + \underbrace{(t - b_i)}_{> r_i} + \underbrace{b_i}_{= r_i}$$

compute v_i's:

$$\overset{2.13}{\Rightarrow} v_i(s - a_i) = v_i(b_i) = r_i$$