Observation: $\forall \sigma \in Aff_2(k)\ \exists!\ \bar\sigma \in Aff_2(L)$ $\boxed{K \subseteq L}$ $\boxed{31.3}$

such that $\bar\sigma(p) = \sigma(p)\ \forall p \in A^2(k)$

(Consider $\sigma = \tau_b \vartheta_A$ for $b \in A^2(k)$, $A \in GL_2(k)$; but $\tau_b \in Aff_2(L)$, ϑ_A — with the same coefficients $\Rightarrow \bar\sigma = \tau_b \vartheta_A$). Cf. §28-10.

$\boxed{TERM}$ $\bar\sigma$ denotes the extension of $\sigma \in Aff_2(k)$ to $Aff_2(L)$ from the observation.

Observation (A): Let $a, f \in k[x,y] \subseteq L[x,y]$ and $\sigma \in Aff_2(k)$

(1) $(\sigma^{-1})^*(a(\sigma^*(x), \sigma^*(y))) = a(\sigma^*(\sigma^{-1})^*(x), \sigma^*(\sigma^{-1})^*(y)) = a(x,y)$,

(2) If $f(x,y) = a(\sigma^*(x), \sigma^*(y)) \overset{(1)}{\Rightarrow} a = (\sigma^{-1})^*(f)$

(3) $(\bar\sigma)^*(a) = \sigma^*(a) \in k[x,y]$

Comment: We need a description of substitution $a(\sigma^*(x), \sigma^*(y))$

Observation (B): Let $\sigma \in Aff_2(k)$, $\tilde\alpha, \tilde\beta \in L$, $u := \sigma^*(x)(\tilde\alpha,\tilde\beta)$, $v := \sigma^*(y)(\tilde\alpha,\tilde\beta)$:

(1) $(u,v) \overset{A(3)}{=} (\bar\sigma^*(x)(\tilde\alpha,\tilde\beta), \bar\sigma^*(y)(\tilde\alpha,\tilde\beta)) = \bar\sigma(\tilde\alpha,\tilde\beta)$,

(2) $(\tilde\alpha,\tilde\beta) \overset{(1)}{=} \bar\sigma^{-1}(u,v) \Rightarrow k(\tilde\alpha,\tilde\beta) = k(u,v)$

(3) $(\sigma^{-1})^*(f)(u,v) = f(\bar\sigma^{-1}(u,v)) \overset{(1)}{=} f(\tilde\alpha,\tilde\beta)$.

Comment: We will find $\sigma \in Aff_2(k)$ for which $(\sigma^{-1})(f)$ will be of the form w

Lemma S7: Let L be an AFF over K given by $f(\alpha,\beta) = 0$, $p = (p_1,p_2) \in A_2(k)$, $A \in GL_2(k)$, ~~σ~~ $\sigma = \vartheta_A \tau_{-p}$, $(u,v) := \bar\sigma(\alpha,\beta)$, $w_\sigma := (\sigma^{-1})^*(f)$. Then:

(1) L is an AFF over K given by $w_\sigma(u,v) = 0$.

(2) If f is smooth at $p \in V_f(k)$, then $\exists A$ such that either $w_\sigma = y$ or $w_\sigma = h(x) + y g(x,y) + y$ where $h \in k[x] \setminus \{0\}$, $g \in k[x,y]$, $mult(h) \geq 2$, $mult\ g \geq 1$

(3) Let $A_p(f) = a_1(x-p_1) + a_2(y-p_2)$ for $p \in V_f(k)$. Then A is a matrix from (2) (i.e. $\sigma = \vartheta_A \tau_{-p}$ satisfies that w_σ is of the form of (2)) $\Leftrightarrow$ $\exists (b_1, b_2) \in k^2 \setminus Span((a_1,a_2))$, $A = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix}$ and.

Proof: (1)(a) $(\sigma^{-1})^*$ is an isomorphism, f is irreducible $\Rightarrow (\sigma^{-1})^*(f)$ is irreducible

(ii) Ob B(3) $\Rightarrow w_\sigma(u,v) = f(\alpha,\beta) = 0$

(iii) Ob B(2) $\Rightarrow k(u,v) = k(\alpha,\beta) = L$

L is an AFF over K given by $w_\sigma(u,v) = 0$

(2)(a) $p \in V_f(k) \Rightarrow \exists a_1, a_2 \in k\ \exists \tilde f \in k[x,y]$ with $mult\ \tilde f \geq 2$ such that

$f(x,y) = a_1(x-p_1) + a_2(y-p_2) + \tilde f(x-p_1, y-p_2)$

(b) §28 $A_p(f) = \tau_p(L(\tau_p^*(f))) = a_1(x-p_1) + a_2(y-p_2)$

(c) f is smooth at $p \Rightarrow A_p(f) \neq 0 \Rightarrow (a_1,a_2) \neq (0,0)$. Put $A = \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$

$A \in GL_2(K) \Rightarrow \Delta = \det A \neq 0 \Rightarrow A^{-1} = \Delta^{-1} \cdot \begin{pmatrix} c_2 & -b_2 \\ -c_1 & b_1 \end{pmatrix}$

Comment: Description of A^{-1} is the key L.A. result

$\sigma = \vartheta_A \tau_{\gamma} \Rightarrow \sigma^{-1} = \tau_{\gamma}^{-1}\vartheta_{A^{-1}} \Rightarrow (\sigma^{-1})^* = (\vartheta_{A^{-1}})^*(\tau_{\gamma})^*$

Com: We will consider when vanish the monomial x and the coefficient of y is 1

$(\sigma^{-1})^*(f) = \vartheta_{A^{-1}}^* \tau_\gamma^*(f) \overset{(a)}{=} \vartheta_{A^{-1}}^*(a_1x + a_2y + \tilde f) =$

$= \Delta^{-1}\cdot(a_1(c_2x - b_2y) + a_2(-c_1x + b_1y)) + \underbrace{\vartheta^*_{A^{-1}}(\tilde f(x,y))}_{\hat f}$! mult $\hat f \geq 2$!

$= \Delta^{-1}\cdot\big(\underbrace{(a_1c_2 - a_2c_1)}_{=0 \Leftrightarrow \exists d:\ (a_1,a_2)=d(c_1,c_2) \wedge d=1} x + (\underbrace{b_1a_2 - b_2a_1}_{=\Delta \Leftrightarrow})y\big) + \hat f \overset{!}{=} y + \hat f \overset{(*)}{\Leftrightarrow} A = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2\end{pmatrix}$, $(0,0) \neq A_\gamma(f)$

As mult $\hat f \geq 2 \Rightarrow \exists\, h \in K[x]$, mult $h \geq 2$, $\exists\, g \in K[x,y]$ mult $g \geq 1$ $\boxed{\hat f = h + yg}$

if $h = 0 \Rightarrow$ ~~u_σ~~ $(\sigma^{-1})^*(f) = y\cdot(1 + g(x,y)) \overset{!}{=} y$
irreducible $\Rightarrow \overset{!}{g = 0}$

(b) follows of (*)!

Theorem 5.8 (corr.) Let L be an AFF over K given by $f(\alpha,\beta) = 0$, f smooth at $\gamma = (\gamma_1,\gamma_2) \in V_f(K)$.

(1) Then $\exists!\, P \in \mathbb{P}_{L|K}$ such that $v_P(\alpha - \gamma_1) > 0$, $v_P(\beta - \gamma_2) \geq 0$

(2) If $l(x,y) = l_0 + l_1x + l_2y \in K[x,y]$ then for P from (1) holds:

$v_P(l(\alpha,\beta))$
- $= 0$ if $l(\gamma) \neq 0$
- $= 1$ if $l(\gamma) = 0$ & $l \notin (A_\gamma(f))$
- ≥ 2 — " — $l \in (A_\gamma(f))$

Comment: K-rational points of the curve V_f are described via places P and tangents are recognized by DV v_P!

Proof. Let $A_\gamma(f) = a_1(x-\gamma_1) + a_2(y - \gamma_2) \neq 0$ as f is smooth at γ.

$\Rightarrow \exists\, (b_1,b_2) \in K^2 \setminus \mathrm{Span}_K(a_1,a_2)$: $A = \begin{pmatrix} b_1 & b_2 \\ a_1 & a_2\end{pmatrix} \in GL_2(K)$, put $\sigma = \vartheta_A \tau_{-\gamma}$

$\overset{5.7}{\Rightarrow} u_\sigma = (\sigma^{-1})^*(f) =$ either y
or $h(x) + y\,g(x,y) + y$ which satisfies the hypotheses of Prop 5.5 for u_σ, u,w; $(u,w) := \sigma(\alpha,\beta)$

if $u_\sigma \neq y \overset{5.5}{\Rightarrow} \exists!\, P \in \mathbb{P}_{L|K}: v_P(u) > 0, v_P(w) > 0$ (& $v_P(w) > v_P(u)$)

if $u_\sigma = y \Rightarrow \beta = 0 \Rightarrow L = K(\alpha) \cong K(x) \Rightarrow \exists!\, P = (\alpha): v_P(\alpha) > 0$

Observe that $\binom{u}{w} = A\binom{\alpha - \gamma_1}{\beta - \gamma_2} \Rightarrow \binom{\alpha-\gamma_1}{\beta-\gamma_2} = A^{-1}\binom{u}{w}$ $\Rightarrow v_P(\alpha) = 1$, $v_P(\beta) = v_P(0) = \infty$

$\Rightarrow \mathrm{Span}_K(u,w) = \mathrm{Span}_K(\alpha - \gamma_1, \beta - \gamma_2) \Rightarrow l(\alpha,\beta) - l(\gamma) = \underbrace{l_1(\alpha - \gamma_1) + l_2(\beta - \gamma_2)}_{\in \mathrm{Span}_K(u,w)}$

As $l(\alpha,\beta) - l(\gamma) \in \mathrm{Span}(u,w) \subseteq P$: $v_P(l(\alpha,\beta)) > 0 \Leftrightarrow l(\alpha,\beta) \in P \Leftrightarrow l(\gamma_1,\gamma_2) \in P$
$\Leftrightarrow l(\gamma) = 0$ $(P \cap K)$

$\Rightarrow \boxed{v_P(l(\alpha,\beta)) = 0 \text{ if } l(\gamma) \neq 0}$

Let $l(\gamma) = 0$ in the rest of the proof $\Rightarrow l(\alpha,\beta) = l_1(\alpha - \gamma_1) + l_2(\beta - \gamma_2) \in \mathrm{Span}_K(u,w)$

$v_P(u) = 1$, $v_P(w) \geq 1$ $\overset{2.13}{\Rightarrow} v_P(l(\alpha,\beta)) > 1 \Leftrightarrow l_1(\alpha - \gamma_1) + l_2(\beta - \gamma_2) \in \mathrm{Span}(w) \Leftrightarrow$

$\Leftrightarrow \exists\, c \in K: (l_1,l_2)A^{-1} = (0,c) \Leftrightarrow (l_1,l_2)\binom{a_2}{-a_1} = 0$ ← 1st column of ΔA^{-1}

Com: The key role plays in the proof L.A.

$\Leftrightarrow \exists\, d \in K: (l_1,l_2) = d(a_1,a_2) \Leftrightarrow l \in (A_\gamma(f))$