K - is a field, we have defined the field $K(C)$ as the fraction field of $K[C]: K[x,y]/(w)$ where $C = V_w$.

Observation: Let $a \in K[x,y]$ be irreducible, $f \in K[x,y]$, $C = V_a$.

(1) $f(x+(a), y+(a)) = 0$ in $K(C) \iff f \in (a)$

(2) $K(C) =$ the fraction field of $K[x,y]/(a) = K(x+(a), y+(a))$

(3) $x+(a)$ is algebraic over $K \overset{(1)}{\iff} \exists p \in K[x] - \{0\} : p(x) \in (a) \iff$
$\iff a | p \iff \deg_y a(x,y) = 0$

Lemma 4.6 Let $K \subseteq L$ be a field extension, $w \in K[x,y]$ irreducible and $\alpha, \beta \in L$: α is transcendental over K, $L = K(\alpha, \beta)$ and $w(\alpha,\beta) = 0$.
Then $[L : K(\alpha)] = \deg_y(w(x,y))$.

Comment: This is an analogue of the claim about simple extension "$[K(\alpha):K] = \deg m_{\alpha,K}$ ← minimal polynomial", i.e. irreducible polynomial w defining AFF plays a similar role as minimal polynomials

Proof: since α is transcendental & $w \neq 0$, $w(\alpha, y) \neq 0$
define $m(y) := w(\alpha, y) \in K(\alpha)[y] - \{0\} \Rightarrow m(\beta) = w(\alpha,\beta) = 0$ (i.e. β is a root of m)
w irreducible $\in K[x,y] \overset{\text{Gauss le}}{\Rightarrow} w$ is irreducible $\in K(x)[y] \overset{\alpha\text{-transcend.}}{\Rightarrow} m(y) \in K(\alpha)[x]$ is irreducible
$\deg_y w > 0$ (as $w(\alpha,\beta) = 0$) We repeat argument from the Observation before L. 4.7

β algebraic over $K(\alpha)$ & $m(y)$ || minimal polynomial of β

$[L : K(\alpha)] = [K(\alpha,\beta) : K(\alpha)] = [K(\alpha)(\beta) : K(\alpha)] = \deg_y m(y) = \deg_y w$

Proposition 4.7 Let $w \in K[x,y]$ be irreducible, $C = V_w$ (i.e. C is an irreducible affine planar curve), put $\alpha := x+(w), \beta := y+(w) \in K[C] \subseteq K(C) \overset{\text{Observation (2)}}{=} K(\alpha,\beta)$

Then: (1) α is transcendental over $K \iff \deg_y(w) > 0$

(2) —— " —— $\Rightarrow [K(C) : K(\alpha)] = \deg_y w$

(3) $K(C)$ is an AFF over K

Proof: (1) follows immediately from Observation (3) (by negation of the both sides)

(2) Observation (1) $\Rightarrow w(\alpha,\beta) = 0$, α - transcendental $\} \overset{L.4.6.}{\Longrightarrow} [\underset{K(\alpha,\beta)}{K(C)} : K(\alpha)] = \deg_y w$

(3) w irreducible $\Rightarrow w \in K[x,y] - K \Rightarrow$ either $\deg_x w > 0$ or $\deg_y w > 0$
w.l.o.g. $\deg_y w > 0 \overset{(1)}{\Rightarrow} \alpha$ is transcendental over $K \overset{(2)}{\Rightarrow} [K(C) : K(\alpha)] < \infty$
$\Rightarrow K(C)$ is a AFF

Corollary 4.8: Let $K \subseteq L$ be a field extension. Then $\exists \alpha,\beta \in L : L = K(\alpha,\beta)$ is an AFF over $K \iff \exists$ irreducible affine curve $C \subseteq \mathbb{A}^2$ s.t. $L \cong_K K(C)$

Comment: AFF's "are" exactly function fields of planar curves

Proof ($\Leftarrow$) by 4.7, ($\Rightarrow$) Let $L = K(\alpha,\beta)$ be AFF, α-transcendental, $[L:K(\alpha)] < \infty$

$\Omega : K[x,y] \to K[\alpha,\beta]$ - substitution, $h(x,y) \to h(\alpha,\beta)$
$\Rightarrow \Omega$ is a homomorphism onto $K[\alpha,\beta]$
$K[x,y]/\ker \Omega \cong K[\alpha,\beta]$ by the 1st Isomorphism Theorem

$K[\alpha,\beta]$ is a domain $\Rightarrow$ $\ker \Omega$ is a prime ideal, not maximal

Using classification of primes in 4.4 we get $\ker \Omega = (w)$ for a prime ideal w

Put $C := V_w$ (an irred. affine curve): $K[C] \cong_K K[\alpha,\beta]$, $a+(w) \to a(\alpha,\beta)$ $\Rightarrow K(C) \cong_K K(\alpha,\beta)$

$\boxed{\text{T\&N}}$ $f\in K[x,y]$ is said to be absolutely irreducible if f is irreducible in $\bar K[x,y]$ (where $\bar K$ denotes the algebraic closure of K)

Lemma 4.9 Let $f,g\in K[x]$, $\deg g\le 1$, $\deg f\ge 3$ is odd. Then $w = y^2+yg-f\in K[x,y]$ is absolutely irreducible. In particular, each WEP is absolutely irreducible. Comment: Thus V_w for w a WEP is irreducible affine curve.

proof: Let $w = u\cdot v\in \bar K[x,y]$, Note $\deg_y w = 2$

(a) Assume that $u,v\in \bar K[x,y]\setminus \bar K[x]$ $\Rightarrow$ $\deg_y u>0$, $\deg_y v>0$

& $2=\deg_y w = \underbrace{\deg_y u}_{>0}+\underbrace{\deg_y v}_{>0}$ $=$ —"— $=1$, —"— $=1$

w.l.o.g.: u & v are monic since $\mathrm{lc}_y(w) = \mathrm{lc}_y(u)\cdot \mathrm{lc}_y(v)$ "leading coefficient"

$\Rightarrow \exists\, s_1,s_2\in\bar K[x]$: $u = y-s_1$, $v=y-s_2$ $\Rightarrow$ $w = y^2 \underbrace{-(s_1+s_2)}_{=g} y + \overbrace{s_1s_2}^{-f}$

$\deg f = \deg s_1s_2 = \deg s_1+\deg s_2$ is odd $\Rightarrow$ $\deg s_1 \ne \deg s_2$

w.l.o.g. $\deg s_1<\deg s_2$ $\Rightarrow$ $\deg(-s_1-s_2) = \deg(s_2) = \deg(g)\le 1$

$\Rightarrow$ $\deg f = \deg s_1s_2 = \underbrace{\deg s_1}_{<1}+\underbrace{\deg s_2}_{\le 1} \le 1$, a contradiction note $\deg f\ge 3$

(b) w.l.o.g. $u\in \bar K[x]$ $\Rightarrow$ $1 = \mathrm{lc}_y(w) = \underbrace{\mathrm{lc}_y(u)}_{=u}\cdot \mathrm{lc}_y(v)$ $\Rightarrow$ $u\in\bar K^*$

$\Rightarrow$ w is irreducible over $\bar K[x,y]$

Lemma 4.10: Let $w\in K[x,y]$ be irreducible, $C=V_w$ & $\tilde K$ be the field of constants of AFF $K(C)$ over K

Then: (1) $K=\tilde K$ $\Leftrightarrow$ w is irreducible in $\tilde K[x,y]$

(2) w is absolute irreducible $\Rightarrow$ $K=\tilde K$

Comment: AFF $K(V_w)$ given by absolutely irreducible polynomial w has all $\alpha\in K(V_w)\setminus K$ transcendental over K!

proof: (1)($\Rightarrow$) w is irreducible in $K[x,y]=\tilde K[x,y]$ ✓

($\Leftarrow$) $\tilde K$ - algebraic closure of K in $K(C)$

$K(C)$ is an AFF $\Rightarrow$ $\exists\,\alpha\in K(C)$ transcendental over K $\Rightarrow$ α transcendental over $\tilde K$

4.8 & 4.7 $\exists\,\beta\in K(C)$: $K(C)=K(\alpha,\beta)$, $w(\alpha,\beta)=0$

$\tilde K\subseteq K(\alpha,\beta)$ $\Rightarrow$ $\tilde K(\alpha,\beta) = K(\alpha,\beta)$, w irreducible over $\tilde K$ ($\Rightarrow$ over $\tilde K(\alpha)$) $\}$ $\Rightarrow$ $w(\alpha,\beta)=0$

4.6. $\Rightarrow$ $[\tilde K(\alpha,\beta):\tilde K(\alpha)] \overset{4.6}{=} \deg_y w \overset{4.6}{=} [K(\alpha,\beta):K(\alpha)]$ $\Rightarrow$ $\dim_{K(\alpha)} K(C) = \dim_{\tilde K(\alpha)} K(C)$

$\Rightarrow K(\alpha)=\tilde K(\alpha)$ $\overset{\alpha\text{-transcend.}}{\Rightarrow}$ $K(x)=\tilde K(x)$ $\Rightarrow$ $[\tilde K(x):K(x)]=1$ $\overset{1.8(2)}{\Rightarrow}$ $K=\tilde K$

(2) w irreducible over $\bar K$ $\Rightarrow$ w irred. over $\tilde K$ $\overset{(1)}{\Rightarrow}$ $K=\tilde K$