Definition. Let X be a set and $\mathcal{T}$ be a family of subsets of X. We say that $(X, \mathcal{T})$ is a *topological space* if the following conditions are satisfied

(a) $\emptyset$ and X belong to $\mathcal{T}$,
(b) if $G_\alpha \in \mathcal{T}$ for every $\alpha \in I$, then $\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$
(c) if $G_i \in \mathcal{T}$ for every $i \in \{1, \dots, n\}, n \in \mathbf{N}$, then $\bigcap_{i=1}^n G_i \in \mathcal{T}$.

The sets from $\mathcal{T}$ are called **open** and their complements are called **closed**.

Remark. Let (M, d) be a metric space. Let $\mathcal{T}$ be the family of all open sets in (M, d) in the sense of the theory of metric spaces. Then the pair $(M, \mathcal{T})$ is a topological space. We say that the topology $\mathcal{T}$ is generated by the metric d. Since each normed linear space can be considered also as a metric space, the preceding remark shows that each normed linear space can be considered also as a topological space. The topology which is defined using the given norm is called **norm topology**.

A topological space $(X, \mathcal{T})$ is called **metrizable** if there exists a metric ϱ on X such that the topology generated by the metric ϱ is $\mathcal{T}$.

Theorem (properties of closed sets)**.** *Let $(X, \mathcal{T})$ be a topological space.*

(a) *The sets $\emptyset$ and X are closed in $(X, \mathcal{T})$,*
(b) *if F_α is closed in $(X, \mathcal{T})$ for every $\alpha \in I$, then $\bigcap_{\alpha \in I} F_\alpha$ is closed in $(X, \mathcal{T})$,*
(c) *if F_i is closed for every $i \in \{1, \dots, n\}, n \in \mathbf{N}$, then $\bigcup_{i=1}^n F_i \in \mathcal{T}$.*

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The **closure** of A is the intersection of all closed sets containing A, i.e., the smallest closed set containing A. The **interior** of A is the union of all open sets contained in A, i.e., the biggest open set contained in A.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say that $(X, \mathcal{T})$ is

(a) T_1 if for every $x, y \in X, x \neq y$, there exists an open set G such that $x \in G$ and $y \notin G$,
(b) T_2 or **Hausdorff** topological space if for every $x, y \in X, x \neq y$, there exist open sets G_1, G_2 such that $x \in G_1, y \in G_2$, and $G_1 \cap G_2 \neq \emptyset$.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces and $f\colon X \to Y$ be a mapping. We say that f is a **continuous** mapping from $(X, \mathcal{T})$ to $(Y, \mathcal{S})$ if $f^{-1}(G)$ is open in $(X, \mathcal{T})$ whenever G is open in $(Y, \mathcal{S})$.

Remark. The above notion extends the notion of continuity from metric space setting to topological one.

Definition. Let $(X, \mathcal{T})$ be a topological space. We say that a family $\mathcal{B} \subset \mathcal{T}$ is a **basis** of $(X, \mathcal{T})$ if any $G \in \mathcal{T}$ is a union of elements from $\mathcal{B}$. We say that a family $\mathcal{S} \subset \mathcal{T}$ is a **subbasis** of $(X, \mathcal{T})$ if the family of finite intersections of elements of $\mathcal{S}$ forms a basis of $(X, \mathcal{T})$.

Remark. Let X be a set and $\mathcal{S}$ be a family of subset of X. Then the topology generated by $\mathcal{S}$ is the smallest topology containing the family $\mathcal{S}$, i.e., this is the topology $\mathcal{T}$ such that $G \in \mathcal{T}$ if and only if G is a union of finite intersections of elements of $\mathcal{S}$. Thus $\mathcal{S}$ is a subbasis of $\mathcal{T}$.

Definition. Let I be a nonempty set, X be a set, and $\mathcal{F}$ be a family of mappings f_α of X into a topological space $(Y_\alpha, \mathcal{T}_\alpha)$, $\alpha \in I$. The **(projective) topology** generated by the family $\mathcal{F}$ is the smallest topology $\mathcal{T}$ such that each mapping f_α is a continuous mapping from $(X, \mathcal{T})$ to $(Y_\alpha, \mathcal{T}_\alpha)$, i.e. $\mathcal{T}$ is the topology generated by the family of sets

$$\{f_\alpha^{-1}(G);\ \alpha \in I, G \in \mathcal{T}_\alpha\}.$$

Remark. (a) Let I be a nonempty set and (X_α, T_α) be a topological space for every $\alpha \in I$. Set $X = \prod_{\alpha \in I} X_\alpha$. The **product topology** on X is the topology generated by canonical projections from X to X_α. A basis of this topology is formed by the sets $\prod_{\alpha \in I} G_\alpha$, where each $G_\alpha \in \mathcal{T}_\alpha$ and for all α but finitely many we have $G_\alpha = X_\alpha$.

(b) Let X be a normed linear space. Then the **weak topology** is generated by the family of linear functionals which are continuous with respect to the norm topology.

Definition. We say that a topological space $(X, \mathcal{T})$ is **compact** if for every family of open sets $\mathcal{G}$ which covers X, i.e., $X = \bigcup \mathcal{G}$, there exists a finite family $\mathcal{G}_0 \subset \mathcal{G}$ which covers X.

Theorem (Tychonoff's theorem)**.** *Let I be a nonempty set and $(X_\alpha, \mathcal{T}_\alpha)$ be a compact topological space for every $\alpha \in I$. Then the space $\prod_{\alpha \in I} X_\alpha$ with the product topology is compact.*