Functional analysis 1

Winter semester 2013-14

1. Topological vector spaces

Basic notions.

Notation. (a) The symbol $\mathbb{F}$ stands for the set of all reals or for the set of all complex numbers.

(b) Let (X,τ) be a topological space and $x \in X$. An open set G containing x is called **neighborhood** of x. We denote $\tau(x) = \{G \in \tau;\ x \in G\}$.

Definition. Suppose that τ is a topology on a vector space X over $\mathbb{F}$ such that

- (X,τ) is T_1, i.e., $\{x\}$ is a closed set for every $x \in X$, and
- the vector space operations are continuous with respect to τ, i.e., $+\colon X \times X \to X$ and $\cdot\colon \mathbb{F} \times X \to X$ are continuous.

Under these conditions, τ is said to be a **vector topology** on X and $(X,+,\cdot,\tau)$ is a **topological vector space** (TVS).

Remark. Let X be a TVS.

(a) For every $a \in X$ the mapping $x \mapsto x+a$ is a homeomorphism of X onto X.
(b) For every $\lambda \in \mathbb{F} \setminus \{0\}$ the mapping $x \mapsto \lambda x$ is a homeomorphism of X onto X.

Definition. Let X be a vector space over $\mathbb{F}$. We say that $A \subset X$ is

- **balanced** if for every $\alpha \in \mathbb{F}$, $|\alpha| \le 1$, we have $\alpha A \subset A$,
- **absorbing** if for every $x \in X$ there exists $t \in \mathbf{R}, t > 0$, such that $x \in tA$,
- **symmetric** if $A = -A$.

Definition. Let X be a TVS and $A \subset X$. We say that A is **bounded** if for every $V \in \tau(0)$ there exists $s > 0$ such that for every $t > s$ we have $A \subset tV$.

Definition. We say that a TVS space X is

- **locally convex** if there exists a basis of 0 whose members are convex,
- **locally bounded** if 0 has a bounded neighborhood,
- **metrizable** if its topology is compatible with some metric on X,
- **F-space** if its topology is induced by a complete invariant metric,
- **Fréchet space** if X is a locally convex F-space,
- **normable** if a norm exists on X such that the metric induced by the norm is compatible with the topology on X.

Theorem 1.1. *Let (X,τ) be a TVS.*

(a) *If $K \subset X$ is compact, $C \subset X$ is closed, and $K \cap C = \emptyset$, then there exists $V \in \tau(0)$ such that $(K+V) \cap (C+V) = \emptyset$.*
(b) *For every neighborhood $U \in \tau(0)$ there exists $V \in \tau(0)$ such that $\overline{V} \subset U$.*
(c) *The space X is a Hausdorff space, i.e., for every $x_1, x_2 \in X, x_1 \neq x_2$, there exist disjoint open sets G_1, G_2 such that $x_i \in G_i, i = 1,2$.*

Theorem 1.2. *Let X be a TVS, $A \subset X$, and $B \subset X$. Then we have*

(a) $\overline{A} = \bigcap\{A+V;\ V \in \tau(0)\}$,
(b) $\overline{A} + \overline{B} \subset \overline{A+B}$,
(c) *if V is a vector subspace of X, then $\overline{V}$ is a vector subspace of X,*
(d) *if A is convex, then $\overline{A}$ and $\operatorname{int} A$ are convex,*
(e) *if A is balanced, then $\overline{A}$ is balanced; if moreover $0 \in \operatorname{int} A$, then $\operatorname{int} A$ is balanced,*
(f) *if A is bounded, then $\overline{A}$ is bounded.*

Theorem 1.3. *Let X be a TVS.*

(a) *For every $U \in \tau(0)$ there exists balanced $V \in \tau(0)$ with $V \subset U$.*
(b) *For every convex $U \in \tau(0)$ there exists balanced convex $V \in \tau(0)$ with $V \subset U$.*

Corollary 1.4. *Let X be a TVS.*

(a) *The space X has a balanced local base.*
(b) *If X is locally convex, then it has a balanced convex local base.*

Theorem 1.5. *Let (X, τ) be a TVS and $V \in \tau(0)$.*

(a) *If $0 < r_1 < r_2 < \dots$ and $\lim r_n = \infty$, then $X = \bigcup_{n=1}^{\infty} r_n V$.*
(b) *Every compact subset $K \subset X$ is bounded.*
(c) *If $\delta_1 > \delta_2 > \delta_3 > \dots$, $\lim \delta_n = 0$, and V is bounded, then the collection $\{\delta_n V;\ n \in \mathbf{N}\}$ is a local base for X.*

Linear mappings.

Theorem 1.6. *Let (X, τ) and (Y, σ) be TVS and $T\colon X \to Y$ be a linear mapping. Then the following are equivalent.*

(i) *T is continuous.*
(ii) *T is continuous at 0.*
(iii) *T is* ***uniformly continuous****, i.e., for every $U \in \sigma(0)$ there exists $V \in \sigma(0)$ such that for every $x_1, x_2 \in X$ with $x_1 - x_2 \in V$ we have $T(x_1) - T(x_2) \in U$.*

Theorem 1.7. *Let $T\colon X \to \mathbb{F}$ be a nonzero linear mapping. Then the following are equivalent.*

(i) *T is continuous.*
(ii) *$\ker T$ is closed.*
(iii) *$\overline{\ker T} \neq X$.*
(iv) *T is bounded on some $V \in \tau(0)$.*

Metrization.

Theorem 1.8. *Let X be a TVS with a countable local base. Then there is a metric d on X such that*

(a) *d is compatible with the topology of X,*
(b) *the open balls centered at 0 are balanced,*
(c) *d is invariant.*

If, in addition, X is locally convex, then d can be chosen so as to satisfy (a), (b), (c), and also

(d) *all open balls are convex.*

Corollary 1.9. *Let X be a TVS. Then the following are equivalent.*

(i) *X is metrizable.*
(ii) *X is metrizable by an invariant metric.*
(iii) *X has a countable local base.*

Theorem 1.10. *(a) If d is an invariant metric on a vector space X then $d(nx, 0) \leq nd(x, 0)$ for every $x \in X$ and $n \in \mathbf{N}$.*

(b) If $\{x_n\}$ is a sequence in a metrizable topological vector space X and if $\lim x_n = 0$, then there are positive scalars γ_n such that $\lim \gamma_n = \infty$ and $\lim \gamma_n x_n = 0$.

Boundedness and continuity.

Theorem 1.11. *The following two properties of a set E in a topological vector space are equivalent:*

(a) *E is bounded.*
(b) *If $\{x_n\}$ is a sequence in E and $\{\alpha_n\}$ is a sequence of scalars such that $\lim \alpha_n = 0$, then $\lim \alpha_n x_n = 0$.*

Theorem 1.12. *Let X and Y be TVS and $T\colon X \to Y$ be a linear mapping. Consider the following properties.*

(i) *T is continuous.*
(ii) *T is bounded, i.e., $T(A)$ is bounded whenever $A \subset X$ is bounded.*
(iii) *If $\{x_n\}$ converges to 0 in X, then $\{T(x_n);\ n \in \mathbf{N}\}$ is bounded.*
(iv) *If $\{x_n\}$ converges to 0 in X, then $\{T(x_n)\}$ converges to 0.*

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If X is metrizable then the properties* (i)–(iv) *are equivalent.*

Pseudonorms and local convexity.

Definition. (a) A **pseudonorm** on a vector space X is a real-valued function p on X such that

- $\forall x, y \in X\colon p(x+y) \le p(x) + p(y)$ (**subadditivity**),
- $\forall \alpha \in \mathbb{F}\ \forall x \in X\colon p(\alpha x) = |\alpha| p(x)$.

(b) A family $\mathcal{P}$ of pseudonorms on X is said to be **separating** if to each $x \ne 0$ corresponds at least one $p \in \mathcal{P}$ with $p(x) \ne 0$.

(c) Let $A \subset X$ be an absorbing set. The **Minkowski functional** μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0;\ t^{-1}x \in A\}.$$

Theorem 1.13. *Suppose p is a pseudonorm on a vector space X. Then*

(a) $p(0) = 0$,
(b) $\forall x, y \in X\colon |p(x) - p(y)| \le p(x-y)$,
(c) $\forall x \in X\colon p(x) \ge 0$,
(d) $\{x \in X;\ p(x) = 0\}$ *is a subspace,*
(e) *the set $B = \{x \in X;\ p(x) < 1\}$ is convex, balanced, absorbing, and $p = \mu_B$.*

Theorem 1.14. *Let X be a vector space and $A \subset X$ be a convex absorbing set. Then*

(a) $\forall x, y \in X\colon \mu_A(x+y) \le \mu_A(x) + \mu_A(y)$,
(b) $\forall t \ge 0\colon \mu_A(tx) = t\mu_A(x)$,
(c) *μ_A is a pseudonorm if A is balanced,*
(d) *if $B = \{x \in X;\ \mu_A(x) < 1\}$ and $C = \{x \in X;\ \mu_A(x) \le 1\}$, then $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$.*

Theorem 1.15. *Suppose $\mathcal{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathcal{B}$ its Minkowski functional μ_V. Then $\{\mu_V;\ V \in \mathcal{B}\}$ is a separating family of continuous pseudonorms on X.*

Theorem 1.16. *Suppose that $\mathcal{P}$ is a separating family of pseudonorms on a vector space X. Associate to each $p \in \mathcal{P}$ and to each $n \in \mathbf{N}$ the set*

$$V(p,n) = \left\{x \in X;\ p(x) < \tfrac{1}{n}\right\}.$$

Let $\mathcal{B}$ be the collection of all finite intersection of the sets $V(p,n)$. Then $\mathcal{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

(a) *every $p \in \mathcal{P}$ is continuous, and*
(b) *a set $E \subset X$ is bounded if and only if every $p \in \mathcal{P}$ is bounded on E.*

Theorem 1.17. *Let X be a locally convex space with countable local base. Then X is metrizable by an invariant metric.*

Theorem 1.18. *A TVS space X is normable if and only if its origin has a convex bounded neighborhood.*

The Hahn-Banach theorems.

Theorem 1.19. *Suppose that A and B are disjoint, nonempty convex sets in a topological vector space X.*

(a) *If A is open there exist $\Lambda \in X^*$ and $\gamma \in \mathbf{R}$ such that $\operatorname{Re}\Lambda(x) < \gamma \le \operatorname{Re}\Lambda(y)$ for every $x \in A$ and for every $y \in B$.*
(b) *If A is compact, B is closed, and X is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1, \gamma_2 \in \mathbf{R}$, such that $\operatorname{Re}\Lambda(x) < \gamma_1 < \gamma_2 \le \operatorname{Re}\Lambda(y)$ for every $x \in A$ and for every $y \in B$.*

Corollary 1.20. *If X is a locally convex space then X^* separates points on X.*

Theorem 1.21. *Suppose M is a subspace of a locally convex space X, and $x_0 \in X$. If $x_0 \notin \overline{M}$, then there exists $\Lambda \in X^*$ such that $\Lambda(x_0) = 1$ and $\Lambda(x) = 0$ for every $x \in M$.*

Theorem 1.22. *If f is a continuous linear functional on a subspace M of a locally convex space X, then there exists $\Lambda \in X^*$ such that $\Lambda = f$ on M.*

Theorem 1.23. *Suppose B is a closed convex balanced set in a locally convex space X, $x_0 \in X \setminus B$. Then there exists $\Lambda \in X^*$ such that $|\Lambda(x)| \le 1$ for every $x \in B$ and $\Lambda(x_0) > 1$.*

2. Weak topologies

Basic properties.

Definition. Let X be a vector space and M be a subspace of the algebraic dual $X^\sharp$. Denote $\sigma(X, M)$ the topology generated by pseudonorms $x \mapsto |\varphi(x)|$, where $\varphi \in M$.

Lemma 2.1. *Suppose that $\Lambda_1, \dots, \Lambda_n$ and Λ are linear functionals on a vector space X. The following properties are equivalent.*

(i) $\Lambda \in \operatorname{span}\{\Lambda_1, \dots, \Lambda_n\}$
(ii) *There exists $\gamma \in \mathbf{R}$ such that for every $x \in X$ we have*
$$|\Lambda(x)| \le \gamma \max\big\{|\Lambda_i(x)|;\ i \in \{1, \dots, n\}\big\}.$$
(iii) $\bigcap_{i=1}^n \operatorname{Ker}\Lambda_i \subset \operatorname{Ker}\Lambda$

Theorem 2.2. *Suppose X is a vector space and M is a vector subspace of the algebraic dual $X^\sharp$ which is separating. Then $(X, \sigma(X, M))$ is a locally convex space and $(X, \sigma(X, M))^* = M$.*

Definition. Let X be a locally convex space. Then $\sigma(X, X^*)$ is **weak topology** on X and $\sigma(X^*, X)$ is **weak star topology** on X^*.

Theorem 2.3 (Mazur)**.** *Let X be a locally convex space and $A \subset X$ be convex. Then $\overline{A}^w = \overline{A}$.*

Corollary 2.4. *Let X be a locally convex space.*

(a) *A subspace of X is originally closed if and only if it is weakly closed.*
(b) *A convex subset of X is originally dense if and only if it is weakly dense.*

Theorem 2.5. *Suppose X is a metrizable locally convex space. If $\{x_n\}$ is a sequence in X that converges weakly to some $x \in X$, then there is a sequence $\{y_i\}$ in X such that*

(a) *each y_i is a convex combination of finitely many x_n, and*
(b) $\lim y_i = x$ *(with respect to the original topology).*

Polars.

Definition. Let X be a TVS and $A \subset X$. Then the set

$$A^0 = \{x^* \in X^*;\ |x^*(x)| \le 1 \text{ for every } x \in A\}$$

is called **polar** of A. If $A \subset X^*$, then we define

$$A_0 = \{x \in X;\ |x^*(x)| \le 1 \text{ for every } x^* \in A\}.$$

Theorem 2.6 (Banach-Alaoglu)**.** *Let X be a TVS and $V \subset X$ be a neighborhood of 0. Then V^0 is w^*-compact.*

Theorem 2.7 (Bipolar theorem)**.** *Let X be a locally convex space.*

(a) *If $A \subset X$ is a closed convex balanced set, then $(A^0)_0 = A$.*
(b) *If $A \subset X^*$ is w^*-closed convex balanced set, then $A = (A_0)^0$.*

Theorem 2.8 (Goldstin)**.** *Let X be a normed linear space. Then B_X is w^*-dense in $B_{X^{**}}$.*

Theorem 2.9. *Let X be a Banach space. Then X is reflexive if and only if B_X is weakly compact.*

Theorem 2.10. *Let X be a reflexive Banach space and $\{x_n\}$ be a bounded sequence of points from X. Then there exists a weakly convergent subsequence.*

3. Vector integration

Convention. Throughout this section X will stand for a Banach space and (Ω, Σ, μ) will be a finite measure space.

Definition. A function $f\colon \Omega \to X$ is called **simple** if there exist $x_1, \dots, x_n \in X$ and $E_1, \dots, E_n \in \Sigma$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$. A function $f\colon \Omega \to X$ is called **μ-measurable** if there exists a sequence of simple functions $\{f_n\}$ such that $\lim \|f_n(\omega) - f(\omega)\| = 0$ for μ-almost all $\omega \in \Omega$. A function $f\colon \Omega \to X$ is called **weakly μ-measurable** if for each $x^* \in X^*$ the function $x^* \circ f$ is μ-measurable.

Theorem 3.1 (Pettis's measurability theorem)**.** *A function $f\colon \Omega \to X$ is μ-measurable if and only if*

(a) *f is μ-essentially separably valued, i.e., there exists $E \in \Sigma$ with $\mu(E) = 0$ and such that $f(\Omega \setminus E)$ is a norm separable subset of X,*
(b) *f is weakly μ-measurable.*

Corollary 3.2. *A function $f\colon \Omega \to X$ is μ-measurable if and only if f is the μ-almost everywhere uniform limit of a sequence of countably valued μ-measurable functions.*

Definition. A μ-measurable function $f\colon \Omega \to X$ is called **Bochner integrable** if there exists a sequence of simple functions $\{f_n\}$ such that $\lim \int_\Omega \|f_n - f\| d\mu = 0$. In this case, $\int_E f\, \mathrm{d}\mu$ is defined for each $E \in \Sigma$ by $\int_E f\, \mathrm{d}\mu = \lim \int_E f_n\, \mathrm{d}\mu$.

Theorem 3.3. *A μ-measurable function $f\colon \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\| d\mu < \infty$.*

Theorem 3.4. *If f is a μ-Bochner integrable function, then*

(a) $\lim_{\mu(E)\to 0} \int_E f d\mu = 0$,
(b) $\| \int_E f d\mu\| \le \int_E \|f\| d\mu$ *for all $E \in \Sigma$,*
(c) *if $\{E_n\}$ is a sequence of pairwise disjoint members of Σ and $E = \bigcup_{n=1}^\infty E_n$, then*

$$\int_E f d\mu = \sum_{n=1}^\infty \int_{E_n} f d\mu,$$

where the sum on the right is absolutely convergent,

(d) *if* $F(E) = \int_E f d\mu$, *then* F *is of bounded variation and*

$$|F|(E) = \int_E \|f\| d\mu$$

for all $E \in \Sigma$.

Corollary 3.5. *If* f *and* g *are* μ*-Bochner integrable and* $\int_E f \, \mathrm{d}\mu = \int_E g \, \mathrm{d}\mu$ *for each* $E \in \Sigma$, *then* $f = g$ μ*-almost everywhere.*

Theorem 3.6. *Let* Y *be a Banach space,* $T \in \mathcal{L}(X, Y)$ *and* $f : \Omega \to X$ *be* μ*-Bochner integrable. Then* $T \circ f$ *is* μ*-Bochner integrable and* $T(\int_E f \, \mathrm{d}\mu) = \int_E T \circ f \, \mathrm{d}\mu$.

Corollary 3.7. *Let* f *a* g *be* μ*-measurable. If for each* $x^* \in X^*$, $x^* \circ f = x^* \circ g$ μ*-almost everywhere, then* $f = g$ μ*-almost everywhere.*

Corollary 3.8. *Let* f *be* μ*-Bochner integrable. Then for each* $E \in \Sigma$ *with* $\mu(E) > 0$ *one has*

$$\frac{1}{\mu(E)} \int_E f \, \mathrm{d}\mu \in \overline{co}\,(f(E)).$$

4. Banach algebras

Basic properties.

Definition. (a) A **complex algebra** is a vector space A over the complex field $\mathbf{C}$ in which a multiplication is defined that satisfies

- $x(yz) = (xy)z$,
- $(x + y)z = xz + yz$, $x(y + z) = xy + xz$,
- $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all $x, y, z \in A$ and $\alpha \in \mathbf{C}$.

(b) If, in addition, A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\|, \qquad x, y \in A$$

then is called a **Banach algebra**.

(c) If an element $e \in A$ in a Banach algebra satisfies $xe = ex = x$ for every $x \in A$, then e is a **unit element**.

Definition. (a) Suppose A is a complex algebra and φ is a linear functional on A which is not identically 0. If $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$, then φ is called a **complex homomorphism** on A.

(b) An element $x \in A$ is said to be **invertible** if it has an inverse in A, that is, if there exists an element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = e$, where e is the unit element of A.

Theorem 4.1. *If* φ *is a complex homomorphism on a complex algebra* A *with unit* e, *then* $\varphi(e) = 1$, *and* $\varphi(x) \neq 0$ *for every invertible* $x \in A$.

Theorem 4.2. *Suppose that* A *is a Banach algebra with unit,* $x \in A$, $\|x\| < 1$. *Then*

(a) $e - x$ *is invertible,*
(b) $\|(e - x)^{-1} - e - x\| \leq \frac{\|x\|^2}{1 - \|x\|}$,
(c) $|\varphi(x)| < 1$ *for every complex homomorphism* φ *on* A.

Definition. Let A be a Banach algebra with unit.

(a) The set of all invertible elements of A is denoted by $G(A)$.
(b) If $x \in A$, the **spectrum** $\sigma(x)$ of x is the set of all complex numbers λ such that $\lambda e - x$ is not invertible. The complement of $\sigma(x)$ is the **resolvent** set of x.
(c) The **spectral radius** of x is the number $\rho(x) = \sup\{|\lambda|;\ \lambda \in \sigma(x)\}$.

Theorem 4.3. *Suppose* A *is a Banach algebra with unit,* $x \in G(A)$, $h \in A$, $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$. *Then* $x + h \in G(A)$ *and*

$$\|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \leq 2\|x^{-1}\|^3\|h\|^2.$$

Theorem 4.4. *If A is a Banach algebra with unit, then $G(A)$ is an open subset of A and the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.*

Theorem 4.5. *If A is a Banach algebra with unit and $x \in A$, then*

(a) *the spectrum $\sigma(x)$ of x is compact and nonempty, and*
(b) *the spectral radius $\rho(x)$ of x satisfies*

$$\rho(x) = \lim \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}.$$

Theorem 4.6 (Gelfand-Mazur)**.** *If A is a Banach algebra with unit in which every nonzero element is invertible, then A is (isometrically isomorphic to) the field of complex numbers.*

Lemma 4.7. *Suppose V and W are open sets in some topological space X, $V \subset W$, and W contains no boundary point of V. Then V is a union of components of W.*

Lemma 4.8. *Suppose A is a Banach algebra with unit, $x_n \in G(A)$ for every $n \in \mathbf{N}$, x is a boundary point of $G(A)$, and $x_n \to x$ as $n \to \infty$. Then $\|x_n^{-1}\| \to \infty$.*

Theorem 4.9. *(a) If A is a closed subalgebra of a Banach algebra B, and if A contains the unit element of B, then $G(A)$ is a union of components of $A \cap G(B)$.*

(b) Under these conditions, if $x \in A$, then $\sigma_A(x)$ is the union of $\sigma_B(x)$ and a (possibly empty) collection of bounded components of the complement of $\sigma_B(x)$. In particular, the boundary of $\sigma_A(x)$ lies in $\sigma_B(x)$.

Corollary 4.10. *If $\sigma_B(x)$ does not separate $\mathbf{C}$, that is, if its complement Ω_B is connected, then $\sigma_A(x) = \sigma_B(x)$.*

Theorem 4.11. *Suppose A is a Banach algebra with unit, $x \in A$, Ω is an open set in $\mathbf{C}$, and $\sigma(x) \subset \Omega$. Then there exists $\delta > 0$ such that $\sigma(x+y) \subset \Omega$ for every $y \in A$ with $\|y\| < \delta$.*

Holomorphic calculus.

Theorem (Cauchy)**.** *Let $\Omega \subset \mathbf{C}$ be open, $f \in \mathrm{Hol}(\Omega)$ and Γ be a contour in Ω satisfying $\mathrm{ind}_\Gamma \alpha = 0$ for $\alpha \in \mathbf{C} \setminus \Omega$. Then we have*

(a) $f(\lambda)\,\mathrm{ind}_\Gamma \lambda = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-\lambda}\,\mathrm{d}w,\ \lambda \in \Omega \setminus \langle\Gamma\rangle$,
(b) $\int_\Gamma f(w)\,\mathrm{d}w = 0$,
(c) *if Γ_1, Γ_2 are contours in Ω satisfying $\mathrm{ind}_{\Gamma_1} \alpha = \mathrm{ind}_{\Gamma_2} \alpha$ for each $\alpha \in \mathbf{C} \setminus \Omega$, then $\int_{\Gamma_1} f(w)\,\mathrm{d}w = \int_{\Gamma_2} f(w)\,\mathrm{d}w$.*

Theorem. *Let $K \subset \Omega \subset \mathbf{C}$, K be compact and Ω be open. Then there exists a contour Γ in Ω such that*

(a) $\langle\Gamma\rangle \subset \Omega \setminus K$,
(b) $\mathrm{ind}_\Gamma \alpha = \begin{cases} 1, & \alpha \in K, \\ 0, & \alpha \in \mathbf{C} \setminus \Omega. \end{cases}$

Definition. If Γ has the properties (a)–(b) from the previous theorem, then we say that Γ **surrounds** K in Ω.

Notation. Let $K \subset \mathbf{C}$ be compact. Then the symbol $\mathrm{Hol}(K)$ denotes the set of all complex functions which are holomorphic on some open set $\Omega \supset K$.

Notation. Let $x \in A$. Denote $R_\lambda = (\lambda e - x)^{-1},\ \lambda \in \mathbf{C} \setminus \sigma(x)$.

Lemma 4.12. *Let $x, y \in A$.*

(a) *If x commutes with y, then x commutes with R_λ for every $\lambda \in \mathbf{C} \setminus \sigma(y)$.*
(b) *For every $\lambda, \mu \in \mathbf{C} \setminus \sigma(x)$ we have*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda.$$

Theorem 4.13. *Let $x \in A$ a $f \in \mathrm{Hol}(\sigma(x))$. We set*

$$f(x) = \frac{1}{2\pi i}\int_\Gamma f(z)R_z \,\mathrm{d}z,$$

where Γ is a contour surrounding $\sigma(x)$ in $D(f)$. The mapping $\Phi\colon f \mapsto f(x)$ from $\mathrm{Hol}(\sigma(x))$ into A is well-defined and does not depend on the choice of Γ.

Theorem 4.14. *Let $x \in A$ and $f \in \mathrm{Hol}(\sigma(x))$. Then we have*

(a) $(1)(x) = e$ *a* $\mathrm{id}(x) = x$,
(b) Φ *is algebraic homomorphism from* $\mathrm{Hol}(\sigma(x))$ *into* A,
(c) *if* $f_n \in \mathrm{Hol}(D(f))$ *and* $f_n \overset{\mathrm{loc}}{\rightrightarrows} f$ *on* $D(f)$, *then* $f_n(x) \to f(x)$ *in* A,
(d) $f(x)$ *is invertible if and only if* $f \neq 0$ *on* $\sigma(x)$,
(e) $\sigma(f(x)) = f(\sigma(x))$,
(f) $(g \circ f)(x) = g(f(x))$ *pro* $g \in \mathrm{Hol}(\sigma(f(x)))$,
(g) *if* $y \in A$ *commutes with* x, *then* y *commutes with* $f(x)$.

A computation in the proof of 4.14(b).

$$
\begin{aligned}
f(x)g(x) &= -\frac{1}{4\pi^2}\left(\int_\Gamma f(z)R_z\,\mathrm{d}z\right)\left(\int_\Lambda g(w)R_w\,\mathrm{d}w\right)\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(f(z)R_z\left(\int_\Lambda g(w)R_w\,\mathrm{d}w\right)\right)\mathrm{d}z = -\frac{1}{4\pi^2}\int_\Gamma\left(\int_\Lambda f(z)g(w)R_zR_w\,\mathrm{d}w\right)\mathrm{d}z\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(\int_\Lambda f(z)g(w)\frac{R_z-R_w}{w-z}\,\mathrm{d}w\right)\mathrm{d}z\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(\int_\Lambda \frac{f(z)g(w)}{w-z}R_z\,\mathrm{d}w - \int_\Lambda \frac{f(z)g(w)}{w-z}R_w\,\mathrm{d}w\right)\mathrm{d}z\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(f(z)R_z\int_\Lambda\frac{g(w)}{w-z}\,\mathrm{d}w\right)\mathrm{d}z + \frac{1}{4\pi^2}\int_\Gamma\left(\int_\Lambda\frac{f(z)g(w)}{w-z}R_w\,\mathrm{d}w\right)\mathrm{d}z\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(f(z)R_z\int_\Lambda\frac{g(w)}{w-z}\,\mathrm{d}w\right)\mathrm{d}z + \frac{1}{4\pi^2}\int_\Lambda\left(\int_\Gamma\frac{f(z)g(w)}{w-z}R_w\,\mathrm{d}z\right)\mathrm{d}w\\
&= -\frac{1}{4\pi^2}\int_\Gamma\left(f(z)R_z\int_\Lambda\frac{g(w)}{w-z}\,\mathrm{d}w\right)\mathrm{d}z + \frac{1}{4\pi^2}\int_\Lambda\left(g(w)R_w\int_\Gamma\frac{f(z)}{w-z}\,\mathrm{d}z\right)\mathrm{d}w\\
&= \frac{1}{2\pi i}\int_\Gamma f(z)g(z)R_z\,\mathrm{d}z = (fg)(x)
\end{aligned}
$$

Theorem 4.15. *Suppose A is a Banach algebra with unit, $x \in A$, and the spectrum $\sigma(x)$ does not separate 0 from ∞. Then*

(a) *x has a logarithm in A,*
(b) *x has roots of all orders in A.*

5. Gelfand transformation

Definition. A subset J of a commutative complex algebra A is said to be **ideal** if

(a) J is a subspace of A, and
(b) $xy \in J$ whenever $x \in A$ and $y \in J$.

If $J \neq A$, then J is a **proper** ideal. **Maximal** ideals are proper ideals which are not contained in any larger proper ideal.

Theorem 5.1. *(a) If A is a commutative complex algebra with unit, then every proper ideal of A is contained in a maximal ideal of A*

(b) If A is a commutative Banach algebra with unit, then every maximal ideal of A is closed.

Theorem 5.2. *Let A be a commutative Banach algebra with unit. Let Δ be the set of all complex homomorphism of A.*

(a) *Every maximal ideal of A is the kernel of some $h \in \Delta$.*
(b) *If $h \in \Delta$, the kernel of h is a maximal ideal of A.*
(c) *An element $x \in A$ is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta$.*
(d) *An element $x \in A$ is invertible in A if and only if x lies in no proper ideal of A.*
(e) *$\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta$.*

Definition. (a) Let Δ be the set of all complex homomorphisms of a commutative Banach algebra A with unit. The formula $\hat{x}(h) = h(x)$ assigns to each $x \in A$ a function $\hat{x}\colon \Delta \to \mathbf{C}$, we call $\hat{x}$ the **Gelfand transform** of x.

(b) The **Gelfand topology** of Δ is the weakest topology that makes every $\hat{x}$ continuous.

(c) The **radical** of A, denoted by $\operatorname{rad} A$, is the intersection of all maximal ideals of A. If $\operatorname{rad} A = \{0\}$, A is called **semisimple**.

Theorem 5.3. *Let Δ be the maximal ideal space of a commutative Banach algebra A with unit.*

(a) *Δ is a compact Hausdorff space.*
(b) *The Gelfand transform is a homomorphism of A onto a subalgebra $\hat{A}$ of $\mathcal{C}(\Delta)$, whose kernel is $\operatorname{rad} A$. The Gelfand transform is therefore an isomorphism if and only if A is semisimple.*
(c) *For each $x \in A$ we have $\operatorname{Rng} \hat{x} = \sigma(x)$.*

Theorem 5.4. *If $\psi\colon B \to A$ is a homomorphism of a commutative Banach algebra B with unit into a semisimple commutative Banach algebra with unit, then ψ is continuous.*

Lemma 5.5. *If A is a commutative Banach algebra with unit and*

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2}, \qquad s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|},$$

then $s^2 \le r \le s$.

Theorem 5.6. *Suppose A is a commutative Banach algebra with unit.*

(a) *The Gelfand transform is an isometry if and only if $\|x^2\| = \|x\|^2$.*
(b) *A is semisimple and $\hat{A}$ is closed in $\mathcal{C}(\Delta)$ if and only if there exists $K < \infty$ such that $\|x\|^2 \le K\|x^2\|$ for every $x \in A$.*

Definition. A mapping $x \mapsto x^*$ of a complex (not necessarily commutative) algebra A into A is called an **involution** on A if it has the following properties for every $x, y \in A$, and $\lambda \in \mathbf{C}$:

- $(x + y)^* = x^* + y^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$,
- $(xy)^* = y^* x^*$,
- $x^{**} = x$.

Any $x \in A$ for which $x^* = x$ is called **hermitian**, or **self-adjoint**.

Theorem 5.7. *If A is a Banach algebra with unit and an involution, and if $x \in A$, then*

(a) *$x + x^*$, $i(x - x^*)$ and xx^* are hermitian,*
(b) *x has a unique representation $x = u + iv$, with $u \in A$, $v \in A$, and both u and v are hermitian,*
(c) *the unit e is hermitian,*
(d) *x is invertible in A if and only if x^* is invertible, in which case $(x^*)^{-1} = (x^{-1})^*$, and*
(e) *$\lambda \in \sigma(x)$ if and only if $\overline{\lambda} \in \sigma(x^*)$.*

Theorem 5.8. *If a Banach algebra A with unit is commutative and semisimple, then every involution on A is continuous.*

Definition. A Banach algebra A with an involution $x \mapsto x^*$ that satisfies $\|xx^*\| = \|x\|^2$ for every $x \in A$ is called a **C^*-algebra**.

Theorem 5.9 (Gelfand-Naimark)**.** *Suppose A is a commutative C^*-algebra with unit. The Gelfand transform is then an isometric isomorphism of A onto $\mathcal{C}(\Delta)$, which has the additional property $\widehat{x^*} = \overline{\hat{x}}$ for every $x \in A$.*

Theorem 5.10. *If A is a commutative C^*-algebra with unit which contains an element x such that the polynomials in x and x^* are dense in A, then the formula $\widehat{\Psi f} = f \circ \hat{x}$ defines an isometric isomorphism Ψ of $\mathcal{C}(\sigma(x))$ onto A which satisfies $\Psi \overline{f} = (\Psi f)^*$ for every $f \in \mathcal{C}(\sigma(x))$. Moreover, if $f(\lambda) = \lambda$ on $\sigma(x)$, then $\Psi f = x$.*

Definition. Let A be an algebra with an involution. If $x \in A$ and $xx^* = x^*x$, then x is said to be **normal**. A set $S \subset A$ is said to be **normal** if S commutes and if $x^* \in S$ whenever $x \in S$.

Theorem 5.11. *Suppose A is a Banach algebra with an involution, and B is a normal subset of A that is maximal with respect to being normal. Then*

(a) *B is a closed commutative subalgebra of A, and*
(b) *$\sigma_B(x) = \sigma_A(x)$ for every $x \in B$.*

Theorem 5.12. *Every C^*-algebra A has the following properties:*

(a) *Hermitian elements have real spectra.*
(b) *If $x \in A$ is normal, then $\rho(x) = \|x\|$.*
(c) *If $y \in A$, then $\rho(yy^*) = \|y\|^2$.*
(d) *If $u, v \in A$ are hermitian, $\sigma(u) \subset [0, \infty)$, $\sigma(v) \subset [0, \infty)$, then $\sigma(u + v) \subset [0, \infty)$.*
(e) *If $y \in A$, then $\sigma(yy^*) \subset [0, \infty)$.*

Theorem 5.13. *Suppose that A is a C^*-algebra with a unit e, B is a closed subalgebra of A, $e \in B$, and $x^* \in B$ for every $x \in B$. Then $\sigma_A(x) = \sigma_B(x)$ for every $x \in B$.*

6. Operators on Hilbert spaces

In this section the symbol H stands for a nontrivial complex Hilbert space.

Definition. We say that $T \in \mathcal{L}(H)$ is

- **normal**, if $T^*T = TT^*$,
- **selfadjoint** (or also **hermitian**), if $T^* = T$,
- **unitary**, if $T^*T = I = TT^*$,
- **orthogonal projection**, if T is a projection, i.e., $T = T^2$, and $\operatorname{Rng} T \perp \operatorname{Ker} T$.

Lemma 6.1. *Let $T \in \mathcal{L}(H)$. Then*

(a) $\|T^*T\| = \|TT^*\| = \|T\|^2$,
(b) $\operatorname{Ker} T^* = \operatorname{Rng} T^{\perp}$.

Lemma 6.2. *Let $T \in \mathcal{L}(H)$. Then the following are equivalent*

(i) $T = 0$,
(ii) $(Tx, x) = 0$ *for every $x \in H$.*

Corollary 6.3. *Let $S, T \in \mathcal{L}(X)$ for every $x \in H$ satisfy $(Sx, x) = (Tx, x)$. Then $T = S$.*

Theorem 6.4 (characterization of normal operators)**.** *An operator $T \in \mathcal{L}(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for each $x \in H$.*

Theorem 6.5 (properties of normal operators)**.** *Let $T \in \mathcal{L}(H)$ be normal. Then we have*

(a) $\operatorname{Ker} T = \operatorname{Ker} T^*$,
(b) *T is invertible if and only if **bounded from below**, i.e., there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in H$ (Weyl),*
(c) *if $x \in H$ satisfies $Tx = \lambda x$, then $T^*x = \overline{\lambda} x$,*
(d) *if $\lambda_1, \lambda_2 \in \mathbf{C}$ are different eigenvalues of T, then $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$,*
(e) *$\|T^2\| = \|T\|^2$,*
(f) *$\|T\| = \rho(T)$.*

Theorem 6.6 (characterization of selfadjoint operators)**.** *Let $T \in \mathcal{L}(H)$. Then $T = T^*$ if and only if (Tx, x) is a real number for every $x \in H$.*

Theorem 6.7. *Let $S, T \in \mathcal{L}(H)$ and S is selfadjoint. Then $\operatorname{Rng} S \perp \operatorname{Rng} T$ if and only if $ST = 0$.*

Theorem 6.8. *For every $T \in \mathcal{L}(H)$ there exists a unique decomposition $T = S_1 + iS_2$, where S_1, S_2 are selfadjoint operators.*

Definition. Let $T \in \mathcal{L}(H)$. **Numerical range** of the operator T is defined by

$$N(T) = \{(Tx, x);\ x \in S_H\}.$$

Theorem 6.9 (Hilbert–Toeplitz)**.** *Let* $T \in \mathcal{L}(H)$*. Then* $\sigma(T) \subset \overline{N(T)}$*.*

Theorem 6.10 (spectrum of selfadjoint operator)**.** *Let* $T \in \mathcal{L}(H)$ *be selfadjoint. Then* $N(T) \subset \mathbf{R}$ *and if we denote* $m_T = \inf N(T)$*,* $M_T = \sup N(T)$*, then we have*

(i) $\sigma(T) \subset [m_T, M_T]$,
(ii) $\|T\|$ *or* $-\|T\|$ *is in* $\sigma(T)$,
(iii) $m_T, M_T \in \sigma(T)$.

Theorem 6.11 (characterization of unitary operators)**.** *Let* $U \in \mathcal{L}(H)$*. Then the following are equivalent:*

(i) U *is unitary,*
(ii) $\operatorname{Rng} U = H$ *a* $(Ux, Uy) = (x, y)$, $x, y \in H$,
(iii) $\operatorname{Rng} U = H$ *a* $\|Ux\| = \|x\|$, $x \in H$.

Theorem 6.12 (characterization of orthogonal projections)**.** *Let $P \in \mathcal{L}(H)$ be a projection. Then the following are equivalent:*

(i) *P is selfadjoint,*
(ii) *P is normal,*
(iii) *P is orthogonal,*
(iv) *$(Px, x) = \|Px\|^2$, $x \in H$.*

Theorem 6.13 (spectral decomposition of compact normal operator; Hilbert–Schmidt)**.** *Let $T \in \mathcal{L}(H)$ be compact and normal. Then there exists an orthonormal basis of H formed by eigenvectors of T. Further there exist nonzero eigenvalues $\{\lambda_n\}_{n=1}^m$, $m \in \mathbf{N} \cup \{\infty\}$, and an orthonormal basis $\{e_n\}_{n=1}^m$ of the space $\overline{\operatorname{Rng} T}$ such that*

$$Tx = \sum_{n=1}^{m} \lambda_n (x, e_n) e_n, \quad x \in H.$$

7. Spectral decompositions

Continuous calculus.

Theorem 7.1. *Let $T \in \mathcal{L}(H)$ be normal. Then there exists a calculus $\Psi \colon \mathcal{C}(\sigma(T)) \to \mathcal{L}(H)$ with the following properties:*

(1) *$\Psi(p) = \sum_{k,l=0}^n a_{kl} T^k (T^*)^l$ for $p(z) = \sum_{k,l=0}^n a_{kl} z^k \overline{z}^l$,*
(2) *Ψ is algebraic isomorphisms of $\mathcal{L}(H)$, $\Psi(\overline{f}) = (\Psi(f))^*$ and $\|\Psi(f)\|_{\mathcal{L}(H)} = \|f\|_{\mathcal{C}(\sigma(T))}$,*
(3) *$\Psi(f) = f(T)$ for $f \in \operatorname{Hol}(\sigma(T))$,*
(4) *$\sigma(\Psi(f)) = f(\sigma(T))$ for $f \in \mathcal{C}(\sigma(T))$,*
(5) *$\Psi(f)$ is normal for $f \in \mathcal{C}(\sigma(T))$,*
(6) *$\Psi(f)$ is selfadjoint if and only if f is real,*
(7) *if S commutes with T, then S commutes with $\Psi(f)$.*

Borel calculus.

Lemma 7.2 (Lax-Milgram)**.** *Let $B \colon H \times H \to \mathbf{C}$ be linear in the first coordinate and conjugate linear in the second coordinate. Let*

$$M := \sup_{x,y \in B_H} |B(x,y)| < \infty$$

Then there exists a unique $T \in \mathcal{L}(H)$ with $B(x,y) = (Tx, y)$ for $x, y \in H$ and $\|T\| = M$.

Notation. Let P be a metric space, then $\mathcal{B}^b(P)$ denotes the set of all bounded Borel functions from P to $\mathbf{C}$. The set $\mathcal{B}^b(P)$ is equipped by the supremum norm.

Lemma 7.3. *Let P be a compact metric space and $\mathcal{A}$ be the smallest system of complex function on P, which contains continuous functions and is closed with respect to pointwise limit of bounded sequences. Then $\mathcal{A} = \mathcal{B}^b(P)$.*

Theorem 7.4. *Let $T \in \mathcal{L}(H)$ be normal. Then there exists a Borel calculus $\Theta \colon \mathcal{B}^b(\sigma(T)) \to \mathcal{L}(H)$ such that*

(1) *$\Theta = \Psi$ on $\mathcal{C}(\sigma(T))$,*
(2) *if $f_n \in \mathcal{B}^b(\sigma(T))$, $f_n \to f$, and $\{f_n\}$ is bounded, then for every $x, y \in H$ we have $(\Theta(f_n)x, y) \to (\Theta(f)x, y)$,*
(3) *Θ is an algebraic homomorphisms, $(\Theta(f))^* = \Theta(\overline{f})$, $\|\Theta(f)\| \le \|f\|_{\mathcal{B}^b(\sigma(T))}$,*
(4) *$\Theta(f)$ is normal for $f \in \mathcal{B}^b(\sigma(T))$,*
(5) *if $f \in \mathcal{B}^b(\sigma(T))$ is real, then $\Theta(f)$ is selfadjoint,*
(6) *if S commutes with T, then S commutes with $\Theta(f)$ for $f \in \mathcal{B}^b(\sigma(T))$.*

Spectral decomposition of normal operator.

Notation. Let K be a metric space. The system of all Borel subsets of K is denoted by $\mathrm{Borel}(K)$.

Definition. Let K be a nonempty compact metric space. We say that the mapping $E\colon \mathrm{Borel}(K) \to \mathcal{L}(H)$ is **spectral measure**, if we have:

(i) for every $B \in \mathrm{Borel}(K)$ is $E(B)$ an orthogonal projection, $E(\emptyset) = 0$, $E(K) = I$,
(ii) $E(B_1 \cap B_2) = E(B_1)E(B_2)$ for every $B_1, B_2 \in \mathrm{Borel}(K)$,
(iii) $E(B_1 \cup B_2) = E(B_1) + E(B_2)$ for every $B_1, B_2 \in \mathrm{Borel}(K)$ disjoint,
(iv) for every $x \in H$ the mapping $E_{x,x}\colon B \mapsto (E(B)x, x)$ is a measure on K, such that its completion is Radon.

Theorem 7.5. *If $T \in \mathcal{L}(H)$ is normal, then $E\colon \mathrm{Borel}(\sigma(T)) \to \mathcal{L}(H)$ defined as $E(B) = \Theta(\chi_B)$ is a spectral measure and it holds:*

(i) $\forall x \in H\ \forall f \in \mathcal{B}^b(\sigma(T))\colon\ (\Theta(f)x, x) = \int_{\sigma(T)} f \, \mathrm{d}E_{x,x}$,
(ii) *for* $A \in \mathrm{Borel}(\sigma(T))$ *and* $T_A := T|_{\mathrm{Rng}\, E(A)}$ *we have* $T_A \in \mathcal{L}(\mathrm{Rng}\, E(A))$ *and* $\sigma(T_A) \subset \overline{A}$,
(iii) *for every nonempty set* $G \subset \sigma(T)$ *which is open in* $\sigma(T)$ *we have* $E(G) \neq 0$.

Theorem 7.6. *Let $E\colon \mathrm{Borel}(K) \to \mathcal{L}(H)$ be a spectral measure on a nonempty compact metric space K. For every function $f \in \mathcal{B}^b(K)$ there exists a unique $T(f) \in \mathcal{L}(H)$ satisfying $(T(f)x, x) = \int_K f \, \mathrm{d}E_{x,x}$ for every $x \in H$. Further we have*

(i) *the mapping* $T\colon f \mapsto T(f)$ *is linear, multiplicative,* $\|T\| = 1$, *and* $T\big(\overline{f}\big) = \big(T(f)\big)^*$,
(ii) $\|T(f)x\|^2 = \int_K |f|^2 \, \mathrm{d}E_{x,x}$, $x \in H$.

Notation. We denote $T(f) = \int_K f \, \mathrm{d}E = \int_K f(t) \, \mathrm{d}E(t)$.

Theorem 7.7. *Let $T \in \mathcal{L}(H)$ be normal. Then there exists a unique spectral measure E on $\sigma(T)$ such that $T = \int_{\sigma(T)} t \, \mathrm{d}E(t)$.*

Theorem 7.8. *Let $T \in \mathcal{L}(H)$ be normal and $\lambda \in \sigma(T)$. Then we have*

(i) $\mathrm{Rng}\, E(\{\lambda\}) = \mathrm{Ker}(\lambda I - T)$,
(ii) $\lambda \in \sigma_p(T)$ *if and only if* $E(\{\lambda\}) \neq 0$,
(iii) *if λ is an isolated point of $\sigma(T)$, then $\lambda \in \sigma_p(T)$.*

Definition. We say that $T \in \mathcal{L}(H)$ is **positive** if $(Tx, x) \geq 0$ for every $x \in H$. If T is positive we write $T \leq 0$.

Theorem 7.9. *Let $T \in \mathcal{L}(H)$. Then the following are equivalent*

(i) $\forall x \in H\colon (Tx, x) \geq 0$,
(ii) $T = T^*$ *and* $\sigma(T) \subset [0, \infty)$.

Theorem 7.10. *Every positive $T \in \mathcal{L}(H)$ has a unique positive square root $S \in \mathcal{L}(H)$. If T is invertible then S is invertible.*

Theorem 7.11. *If $T \in \mathcal{L}(H)$, then the positive square root of T^*T is the only positive operator $P \in \mathcal{L}(H)$ that satisfies $\|Px\| = \|Tx\|$ for every $x \in H$.*

Theorem 7.12.

(a) *If $T \in \mathcal{L}(H)$ is invertible, then T has a unique* ***polar decomposition*** *$T = UP$, i.e., U is unitary and $P \geq 0$.*
(b) *If $T \in \mathcal{L}(H)$ is normal, then T has a polar decomposition $T = UP$.*