Functional analysis 1

1. Topological vector spaces

Notation

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(b) Let (X, τ) be a topological space and $x \in X$. An open set G containing x is called **neighborhood** of x. We denote $\tau(x) = \{G \in \tau;\ x \in G\}$.

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- (X, τ) is T_1, i.e., $\{x\}$ is a closed set for every $x \in X$, and
- the vector space operations are continuous with respect to τ, i.e., $+ \colon X \times X \to X$ and $\cdot \colon \mathbb{F} \times X \to X$ are continuous.

Under these conditions, τ is said to be a **vector topology** on X and $(X, +, \cdot, \tau)$ is a **topological vector space** (TVS).

Remark

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- **absorbing** if for every $x \in X$ there exists $t \in \mathbf{R}$, $t > 0$, such that $x \in tA$,
- **symmetric** if $A = -A$.

Definition

Let X be a TVS and $A \subset X$. We say that A is **bounded** if for every $V \in \tau(0)$ there exists $s > 0$ such that for every $t > s$ we have $A \subset tV$.

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- **Fréchet space** if X is a locally convex F-space,
- **normable** if a norm exists on X such that the metric induced by the norm is compatible with the topology on X.

Theorem 1.1

Let (X, τ) be a TVS.

(a) *If $K \subset X$ is compact, $C \subset X$ is closed, and $K \cap C = \emptyset$, then there exists $V \in \tau(0)$ such that $(K + V) \cap (C + V) = \emptyset$.*

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(b) *For every neighborhood $U \in \tau(0)$ there exists $V \in \tau(0)$ such that $\overline{V} \subset U$.*

(c) *The space X is a Hausdorff space, i.e., for every $x_1, x_2 \in X, x_1 \neq x_2$, there exist disjoint open sets G_1, G_2 such that $x_i \in G_i, i = 1, 2$.*

Theorem 1.2

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Corollary 1.4

Let X be a TVS.

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Theorem 1.5

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(c) *If $\delta_1 > \delta_2 > \delta_3 > \dots$, $\lim \delta_n = 0$, and V is bounded, then the collection $\{\delta_n V;\ n \in \mathbf{N}\}$ is a local base for X.*

Theorem 1.6

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Let (X, τ) and (Y, σ) be TVS and $T: X \to Y$ be a linear mapping. Then the following are equivalent.

(i) *T is continuous.*

(ii) *T is continuous at 0.*

(iii) *T is* **uniformly continuous**, *i.e., for every $U \in \sigma(0)$ there exists $V \in \sigma(0)$ such that for every $x_1, x_2 \in X$ with $x_1 - x_2 \in V$ we have $T(x_1) - T(x_2) \in U$.*

Theorem 1.7

Let $T \colon X \to \mathbb{F}$ *be a nonzero linear mapping. Then the following are equivalent.*

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(iv) *T is bounded on some $V \in \tau(0)$.*

Theorem 1.8

Let X be a TVS with a countable local base. Then there is a metric d on X such that

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If, in addition, X is locally convex, then d can be chosen so as to satisfy (a), (b), (c), and also

(d) *all open balls are convex.*

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Theorem 1.10

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(b) If $\{x_n\}$ is a sequence in a metrizable topological vector space X and if $\lim x_n = 0$, then there are positive scalars γ_n such that $\lim \gamma_n = \infty$ and $\lim \gamma_n x_n = 0$.

Theorem 1.11

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Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If X is metrizable then the properties* (i)–(iv) *are equivalent.*

Definition

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- $\forall x, y \in X\colon\ p(x+y) \leq p(x) + p(y)$ (**subadditivity**),

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(b) A family $\mathcal{P}$ of pseudonorms on X is said to be **separating** if to each $x \neq 0$ corresponds at least one $p \in \mathcal{P}$ with $p(x) \neq 0$.

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(b) A family $\mathcal{P}$ of pseudonorms on X is said to be **separating** if to each $x \neq 0$ corresponds at least one $p \in \mathcal{P}$ with $p(x) \neq 0$.
(c) Let $A \subset X$ be an absorbing set. The **Minkowski functional** μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0;\ t^{-1}x \in A\}.$$

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(e) *the set* $B = \{x \in X;\ p(x) < 1\}$ *is convex, balanced, absorbing, and* $p = \mu_B$.

Theorem 1.14

Let X be a vector space and $A \subset X$ be a convex absorbing set. Then

(a) $\forall x, y \in X\colon \mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$,

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(d) *if* $B = \{x \in X;\ \mu_A(x) < 1\}$ *and* $C = \{x \in X;\ \mu_A(x) \leq 1\}$, *then* $B \subset A \subset C$ *and* $\mu_A = \mu_B = \mu_C$.

Theorem 1.15

Suppose $\mathcal{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathcal{B}$ its Minkowski functional μ_V. Then $\{\mu_V;\ V \in \mathcal{B}\}$ is a separating family of continuous pseudonorms on X.

Theorem 1.16

Suppose that $\mathcal{P}$ is a separating family of pseudonorms on a vector space X. Associate to each $p \in \mathcal{P}$ and to each $n \in \mathbf{N}$ the set

$$V(p, n) = \left\{x \in X;\ p(x) < \tfrac{1}{n}\right\}.$$

Let $\mathcal{B}$ be the collection of all finite intersection of the sets $V(p, n)$.

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Suppose that $\mathcal{P}$ is a separating family of pseudonorms on a vector space X. Associate to each $p \in \mathcal{P}$ and to each $n \in \mathbf{N}$ the set

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Let $\mathcal{B}$ be the collection of all finite intersection of the sets $V(p,n)$. Then $\mathcal{B}$ is a convex balanced local base for a topology τ on X, which turns X into a locally convex space such that

(a) *every $p \in \mathcal{P}$ is continuous, and*

(b) *a set $E \subset X$ is bounded if and only if every $p \in \mathcal{P}$ is bounded on E.*

Theorem 1.17

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Theorem 1.18

A TVS space X is normable if and only if its origin has a convex bounded neighborhood.

Theorem 1.19

Suppose that A and B are disjoint, nonempty convex sets in a topological vector space X.

(a) *If A is open there exist $\Lambda \in X^*$ and $\gamma \in \mathbf{R}$ such that $\operatorname{Re}\Lambda(x) < \gamma \leq \operatorname{Re}\Lambda(y)$ for every $x \in A$ and for every $y \in B$.*

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(b) *If A is compact, B is closed, and X is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1, \gamma_2 \in \mathbf{R}$, such that* $\operatorname{Re}\Lambda(x) < \gamma_1 < \gamma_2 \leq \operatorname{Re}\Lambda(y)$ *for every $x \in A$ and for every $y \in B$.*

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Suppose that A and B are disjoint, nonempty convex sets in a topological vector space X.

(a) *If A is open there exist $\Lambda \in X^*$ and $\gamma \in \mathbf{R}$ such that $\operatorname{Re} \Lambda(x) < \gamma \leq \operatorname{Re} \Lambda(y)$ for every $x \in A$ and for every $y \in B$.*

(b) *If A is compact, B is closed, and X is locally convex, then there exist $\Lambda \in X^*$, $\gamma_1, \gamma_2 \in \mathbf{R}$, such that $\operatorname{Re} \Lambda(x) < \gamma_1 < \gamma_2 \leq \operatorname{Re} \Lambda(y)$ for every $x \in A$ and for every $y \in B$.*

Corollary 1.20

If X is a locally convex space then X^* separates points on X.

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Theorem 1.21

Suppose M is a subspace of a locally convex space X, and $x_0 \in X$. If $x_0 \notin \overline{M}$, then there exists $\Lambda \in X^*$ such that $\Lambda(x_0) = 1$ and $\Lambda(x) = 0$ for every $x \in M$.

Theorem 1.22

If f is a continuous linear functional on a subspace M of a locally convex space X, then there exists $\Lambda \in X^*$ such that $\Lambda = f$ on M.

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If f is a continuous linear functional on a subspace M of a locally convex space X, then there exists $\Lambda \in X^*$ such that $\Lambda = f$ on M.

Theorem 1.23

Suppose B is a closed convex balanced set in a locally convex space X, $x_0 \in X \setminus B$. Then there exists $\Lambda \in X^*$ such that $|\Lambda(x)| \leq 1$ for every $x \in B$ and $\Lambda(x_0) > 1$.

2. Weak topologies

Definition

Let X be a vector space and M be a subspace of the algebraic dual $X^\sharp$. Denote $\sigma(X, M)$ the topology generated by pseudonorms $x \mapsto |\varphi(x)|$, where $\varphi \in M$.

2. Weak topologies

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Let X be a vector space and M be a subspace of the algebraic dual $X^\sharp$. Denote $\sigma(X, M)$ the topology generated by pseudonorms $x \mapsto |\varphi(x)|$, where $\varphi \in M$.

Lemma 2.1

Suppose that $\Lambda_1, \dots, \Lambda_n$ and Λ are linear functionals on a vector space X. The following properties are equivalent.

(i) $\Lambda \in \operatorname{span}\{\Lambda_1, \dots, \Lambda_n\}$

(ii) *There exists $\gamma \in \mathbf{R}$ such that for every $x \in X$ we have*

$$|\Lambda(x)| \leq \gamma \max\big\{|\Lambda_i(x)|;\ i \in \{1, \dots, n\}\big\}.$$

(iii) $\bigcap_{i=1}^{n} \operatorname{Ker} \Lambda_i \subset \operatorname{Ker} \Lambda$

2. Weak topologies

Theorem 2.2

Suppose X is a vector space and M is a vector subspace of the algebraic dual $X^\sharp$ which is separating. Then $(X, \sigma(X, M))$ is a locally convex space and $(X, \sigma(X, M))^* = M$.

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Definition

Let X be a locally convex space. Then $\sigma(X, X^*)$ is **weak topology** on X and $\sigma(X^*, X)$ is **weak star topology** on X^*.

Theorem 2.3 (Mazur)

Let X be a locally convex space and $A \subset X$ be convex. Then $\overline{A}^w = \overline{A}$.

2. Weak topologies

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Corollary 2.4

Let X be a locally convex space.

(a) *A subspace of X is originally closed if and only if it is weakly closed.*

(b) *A convex subset of X is originally dense if and only if it is weakly dense.*

Theorem 2.5

Suppose X is a metrizable locally convex space. If $\{x_n\}$ is a sequence in X that converges weakly to some $x \in X$, then there is a sequence $\{y_i\}$ in X such that

(a) *each y_i is a convex combination of finitely many x_n, and*

(b) $\lim y_i = x$ *(with respect to the original topology).*

Definition

Let X be a TVS and $A \subset X$. Then the set

$$A^0 = \{x^* \in X^*;\ |x^*(x)| \leq 1 \text{ for every } x \in A\}$$

is called **polar** of A. If $A \subset X^*$, then we define

$$A_0 = \{x \in X;\ |x^*(x)| \leq 1 \text{ for every } x^* \in A\}.$$

Theorem 2.6 (Banach-Alaoglu)

Let X be a TVS and $V \subset X$ be a neighborhood of 0. Then V^0 is w^*-compact.

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Theorem 2.7 (Bipolar theorem)

Let X be a locally convex space.

(a) *If $A \subset X$ is a closed convex balanced set, then $(A^0)_0 = A$.*

(b) *If $A \subset X^*$ is w^*-closed convex balanced set, then $A = (A_0)^0$.*

Theorem 2.8 (Goldstin)

Let X be a normed linear space. Then B_X is w^*-dense in $B_{X^{**}}$.

Theorem 2.9

Let X be a Banach space. Then X is reflexive if and only if B_X is weakly compact.

2. Weak topologies

Theorem 2.10

Let X be a reflexive Banach space and $\{x_n\}$ be a bounded sequence of points from X. Then there exists a weakly convergent subsequence.

3. Vector integration

Convention

Throughout this section X will stand for a Banach space and (Ω, Σ, μ) will be a finite measure space.

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Definition

A function $f : \Omega \to X$ is called **simple** if there exist $x_1, \dots, x_n \in X$ and $E_1, \dots, E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$.

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A function $f : \Omega \to X$ is called μ-**measurable** if there exists a sequence of simple functions $\{f_n\}$ such that $\lim \|f_n(\omega) - f(\omega)\| = 0$ for μ-almost all $\omega \in \Omega$.

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Theorem 3.1 (Pettis's measurability theorem)

A function $f : \Omega \to X$ *is* μ*-measurable if and only if*

(a) f *is* μ*-essentially separably valued, i.e., there exists* $E \in \Sigma$ *with* $\mu(E) = 0$ *and such that* $f(\Omega \setminus E)$ *is a norm separable subset of* X*,*

(b) f *is weakly* μ*-measurable.*

3. Vector integration

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(b) *f is weakly μ-measurable.*

Corollary 3.2

A function $f : \Omega \to X$ is μ-measurable if and only if f is the μ-almost everywhere uniform limit of a sequence of countably valued μ-measurable functions.

Definition

A μ-measurable function $f\colon \Omega \to X$ is called **Bochner integrable** if there exists a sequence of simple functions $\{f_n\}$ such that $\lim \int_\Omega \|f_n - f\| d\mu = 0$. In this case, $\int_E f \, \mathrm{d}\mu$ is defined for each $E \in \Sigma$ by $\int_E f \, \mathrm{d}\mu = \lim \int_E f_n \, \mathrm{d}\mu$.

3. Vector integration

Theorem 3.3

A μ-measurable function $f : \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\| d\mu < \infty$.

Theorem 3.4

If f is a μ-Bochner integrable function, then

(a) $\lim_{\mu(E)\to 0} \int_E f d\mu = 0$,

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(c) *if* $\{E_n\}$ *is a sequence of pairwise disjoint members of* Σ *and* $E = \bigcup_{n=1}^{\infty} E_n$, *then*

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu,$$

where the sum on the right is absolutely convergent,

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where the sum on the right is absolutely convergent,

(d) *if* $F(E) = \int_E f d\mu$, *then* F *is of bounded variation and*

$$|F|(E) = \int_E \|f\| d\mu$$

for all $E \in \Sigma$.

Corollary 3.5

If f and g are μ-Bochner integrable and $\int_E f \, \mathrm{d}\mu = \int_E g \, \mathrm{d}\mu$ for each $E \in \Sigma$, then $f = g$ μ-almost everywhere.

3. Vector integration

Theorem 3.6

Let Y be a Banach space, $T \in \mathcal{L}(X, Y)$ and $f : \Omega \to X$ be μ-Bochner integrable. Then $T \circ f$ is μ-Bochner integrable and $T(\int_E f \, \mathrm{d}\mu) = \int_E T \circ f \, \mathrm{d}\mu$.

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Let f a g be μ-measurable. If for each $x^* \in X^*$, $x^* \circ f = x^* \circ g$ μ-almost everywhere, then $f = g$ μ-almost everywhere.

Corollary 3.8

Let f be μ-Bochner integrable. Then for each $E \in \Sigma$ with $\mu(E) > 0$ one has

$$\frac{1}{\mu(E)} \int_E f \,\mathrm{d}\mu \in \overline{co}(f(E)).$$

4. Banach algebras

Definition

(a) A **complex algebra** is a vector space A over the complex field $\mathbf{C}$ in which a multiplication is defined that satisfies

- $x(yz) = (xy)z$,
- $(x + y)z = xz + yz$, $x(y + z) = xy + xz$,
- $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all $x, y, z \in A$ and $\alpha \in \mathbf{C}$.

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then is called a **Banach algebra**.

(c) If an element $e \in A$ in a Banach algebra satisfies $xe = ex = x$ for every $x \in A$, then e is a **unit element**.

Definition

(a) Suppose A is a complex algebra and φ is a linear functional on A which is not identically 0. If $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$, then φ is called a **complex homomorphism** on A.

Definition

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(b) An element $x \in A$ is said to be **invertible** if it has an inverse in A, that is, if there exists an element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = e$, where e is the unit element of A.

Theorem 4.1

If φ is a complex homomorphism on a complex algebra A with unit e, then $\varphi(e) = 1$, and $\varphi(x) \neq 0$ for every invertible $x \in A$.

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Theorem 4.2

Suppose that A is a Banach algebra with unit, $x \in A$, $\|x\| < 1$. Then

(a) *$e - x$ is invertible,*

(b) $\|(e-x)^{-1} - e - x\| \leq \frac{\|x\|^2}{1-\|x\|}$,

(c) *$|\varphi(x)| < 1$ for every complex homomorphism φ on A.*

Definition

Let A be a Banach algebra with unit.

(a) The set of all invertible elements of A is denoted by $G(A)$.

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(b) If $x \in A$, the **spectrum** $\sigma(x)$ of x is the set of all complex numbers λ such that $\lambda e - x$ is not invertible. The complement of $\sigma(x)$ is the **resolvent** set of x.

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(c) The **spectral radius** of x is the number $\rho(x) = \sup\{|\lambda|;\ \lambda \in \sigma(x)\}$.

Theorem 4.3

Suppose A is a Banach algebra with unit, $x \in G(A)$, $h \in A$, $\|h\| < \frac{1}{2}\|x^{-1}\|^{-1}$. Then $x + h \in G(A)$ and

$$\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \leq 2\|x^{-1}\|^3\|h\|^2.$$

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Theorem 4.4

If A is a Banach algebra with unit, then $G(A)$ is an open subset of A and the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.

Theorem 4.5

If A is a Banach algebra with unit and $x \in A$, then

(a) *the spectrum $\sigma(x)$ of x is compact and nonempty, and*

(b) *the spectral radius $\rho(x)$ of x satisfies*

$$\rho(x) = \lim \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}.$$

Theorem 4.6 (Gelfand-Mazur)

If A is a Banach algebra with unit in which every nonzero element is invertible, then A is (isometrically isomorphic to) the field of complex numbers.

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Suppose V and W are open sets in some topological space X, $V \subset W$, and W contains no boundary point of V. Then V is a union of components of W.

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Lemma 4.8

Suppose A is a Banach algebra with unit, $x_n \in G(A)$ for every $n \in \mathbf{N}$, x is a boundary point of $G(A)$, and $x_n \to x$ as $n \to \infty$. Then $\|x_n^{-1}\| \to \infty$.

Theorem 4.9

(a) If A is a closed subalgebra of a Banach algebra B, and if A contains the unit element of B, then $G(A)$ is a union of components of $A \cap G(B)$.

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Corollary 4.10

If $\sigma_B(x)$ does not separate $\mathbf{C}$, that is, if its complement Ω_B is connected, then $\sigma_A(x) = \sigma_B(x)$.

Theorem 4.11

Suppose A is a Banach algebra with unit, $x \in A$, Ω is an open set in $\mathbf{C}$, and $\sigma(x) \subset \Omega$. Then there exists $\delta > 0$ such that $\sigma(x + y) \subset \Omega$ for every $y \in A$ with $\|y\| < \delta$.

Preliminaries from complex analysis

Theorem (Cauchy)

Let $\Omega \subset \mathbf{C}$ *be open,* $f \in \mathrm{Hol}(\Omega)$ *and* Γ *be a contour in* Ω *satisfying* $\mathrm{ind}_\Gamma \alpha = 0$ *for* $\alpha \in \mathbf{C} \setminus \Omega$. *Then we have*

(a) $f(\lambda)\,\mathrm{ind}_\Gamma \lambda = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-\lambda}\,\mathrm{d}w,\ \lambda \in \Omega \setminus \langle \Gamma \rangle,$

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(b) $\int_\Gamma f(w)\, \mathrm{d}w = 0$,

(c) *if* Γ_1, Γ_2 *are contours in* Ω *satisfying* $\mathrm{ind}_{\Gamma_1}\, \alpha = \mathrm{ind}_{\Gamma_2}\, \alpha$ *for each* $\alpha \in \mathbf{C} \setminus \Omega$, *then* $\int_{\Gamma_1} f(w)\, \mathrm{d}w = \int_{\Gamma_2} f(w)\, \mathrm{d}w$.

Theorem

Let $K \subset \Omega \subset \mathbf{C}$, K be compact and Ω be open. Then there exists a contour Γ in Ω such that

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Definition

If Γ has the properties (a)–(b) from the previous theorem, then we say that Γ **surrounds** K in Ω.

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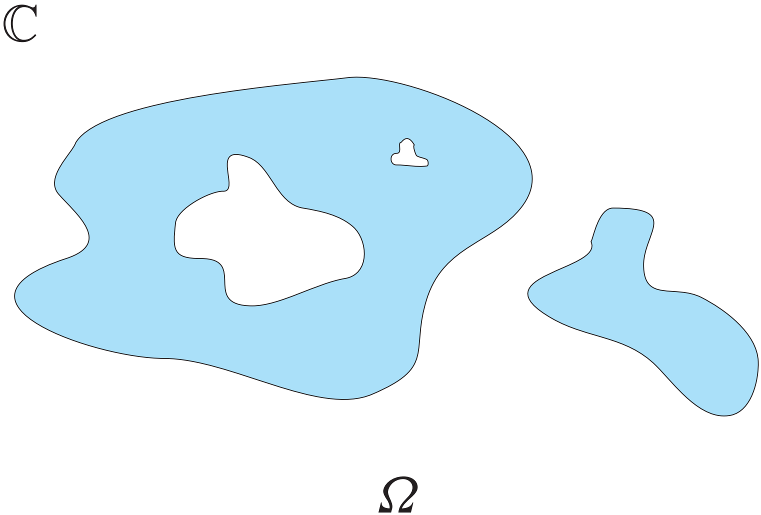
Notation

Let $K \subset \mathbf{C}$ be compact. Then the symbol $\mathrm{Hol}(K)$ denotes the set of all complex functions which are holomorphic on some open set $\Omega \supset K$.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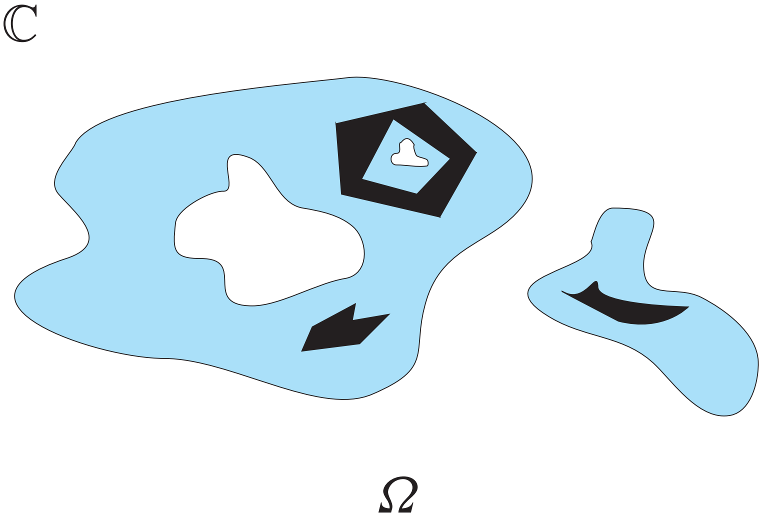
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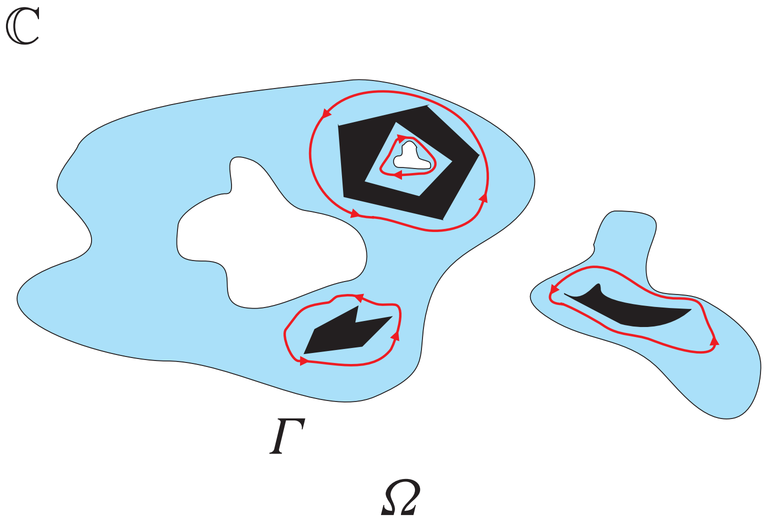
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Let $x \in A$. Denote $R_\lambda = (\lambda e - x)^{-1},\ \lambda \in \mathbf{C} \setminus \sigma(x)$.

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Lemma 4.12

Let $x, y \in A$.

(a) *If x commutes with y, then x commutes with R_λ for every $\lambda \in \mathbf{C} \setminus \sigma(y)$.*

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Lemma 4.12

Let $x, y \in A$.

(a) *If x commutes with y, then x commutes with R_λ for every $\lambda \in \mathbf{C} \setminus \sigma(y)$.*

(b) *For every $\lambda, \mu \in \mathbf{C} \setminus \sigma(x)$ we have*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda.$$

Theorem 4.13

Let $x \in A$ *a* $f \in \mathrm{Hol}(\sigma(x))$*. We set*

$$f(x) = \frac{1}{2\pi i} \int_\Gamma f(z) R_z \, \mathrm{d}z,$$

where Γ *is a contour surrounding* $\sigma(x)$ *in* $D(f)$*.*

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where Γ *is a contour surrounding* $\sigma(x)$ *in* $D(f)$. *The mapping* $\Phi \colon f \mapsto f(x)$ *from* $\mathrm{Hol}(\sigma(x))$ *into* A *is well-defined and does not depend on the choice of* Γ.

Theorem 4.14

Let $x \in A$ *and* $f \in \mathrm{Hol}(\sigma(x))$. *Then we have*

(a) $(1)(x) = e$ *a* $\mathrm{id}(x) = x$,

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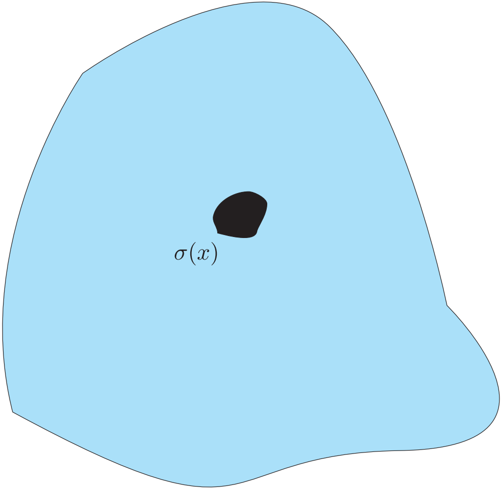
next theorem

Proof of $f(x)g(x) = fg(x)$.



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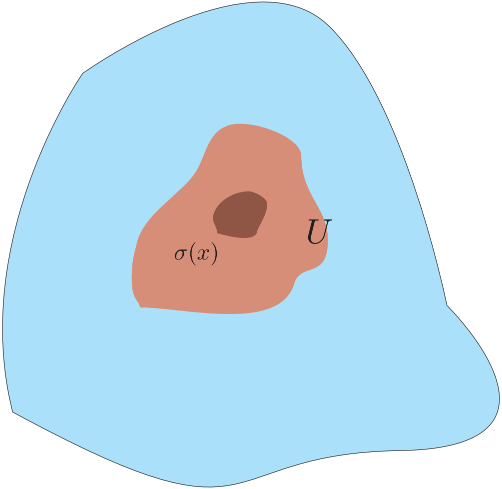

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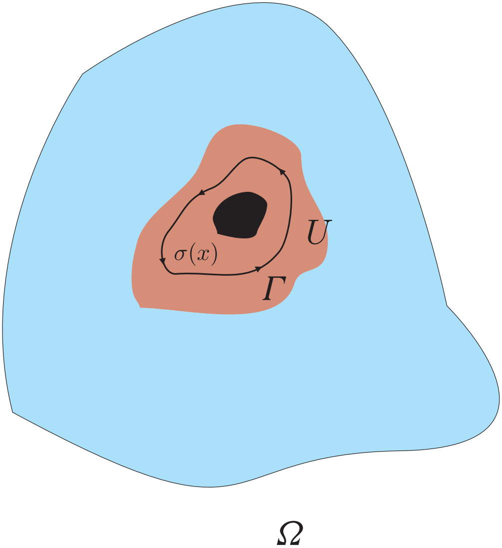

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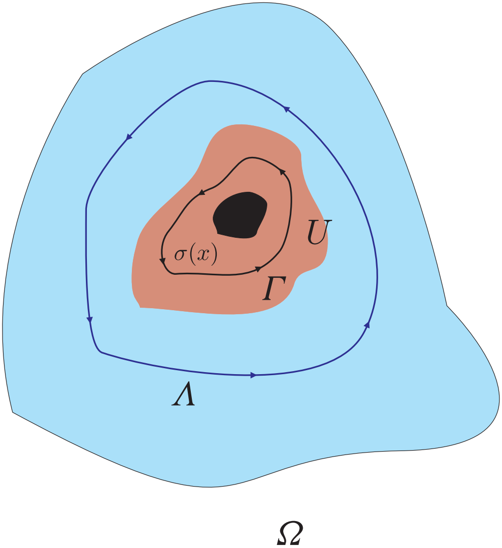

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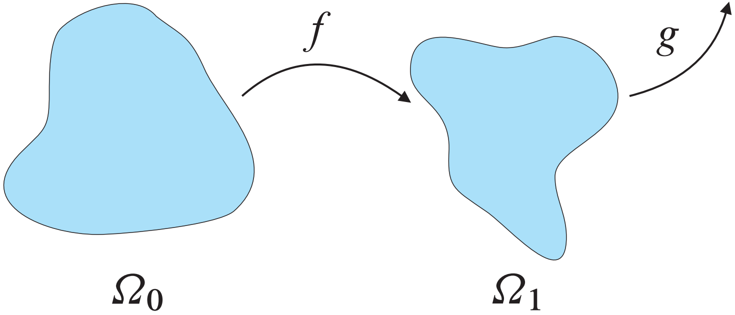
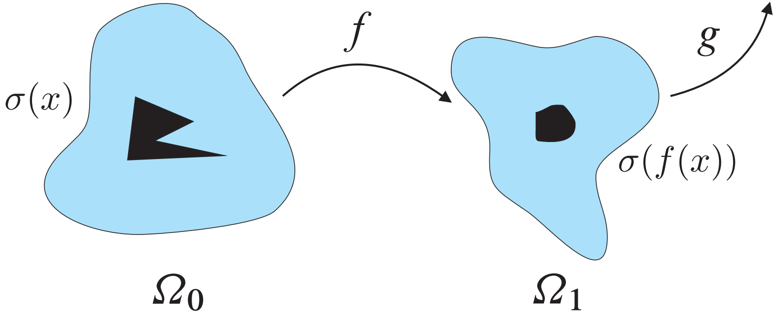


f
g
Ω_0
Ω_1



$\sigma(x)$
f
g
$\sigma(f(x))$
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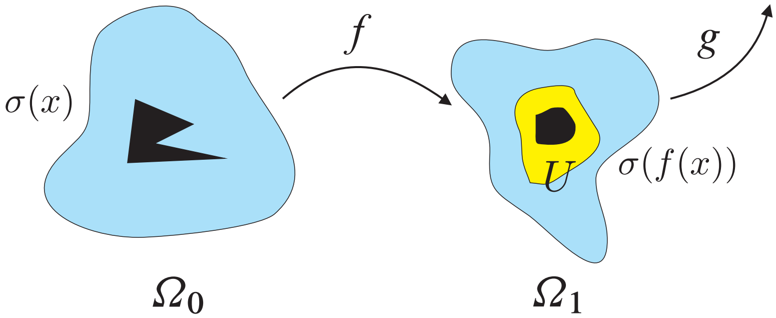

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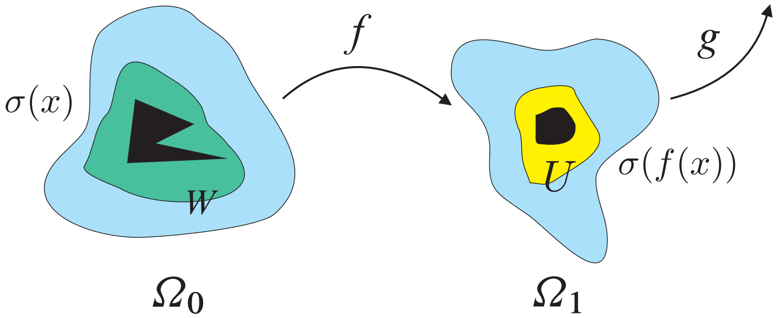
σ(x)
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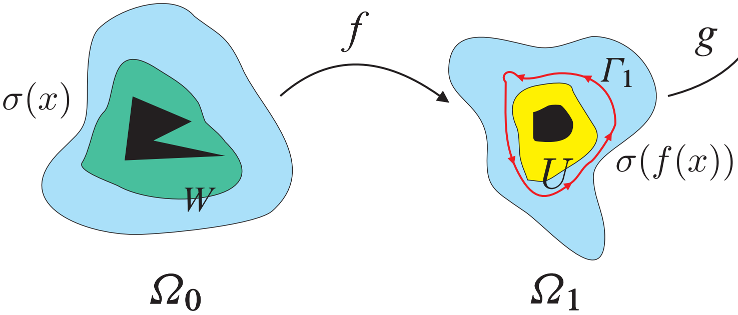

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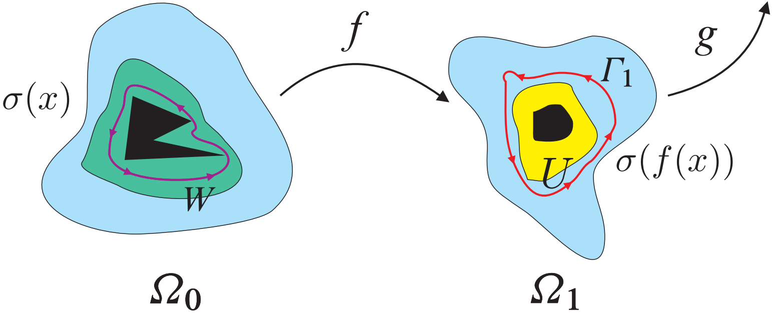

σ(x)
W
f
Γ1
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σ(f(x))
Ω0
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Theorem 4.15

Suppose A is a Banach algebra with unit, $x \in A$, and the spectrum $\sigma(x)$ does not separate 0 from ∞. Then

(a) *x has a logarithm in A,*

(b) *x has roots of all orders in A.*

5. Gelfand transformation

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A subset J of a commutative complex algebra A is said to be **ideal** if

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(b) If A is a commutative Banach algebra with unit, then every maximal ideal of A is closed.

5. Gelfand transformation

Theorem 5.2

Let A be a commutative Banach algebra with unit. Let Δ be the set of all complex homomorphism of A.

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(b) The **Gelfand topology** of Δ is the weakest topology that makes every $\hat{x}$ continuous.
(c) The **radical** of A, denoted by $\operatorname{rad} A$, is the intersection of all maximal ideals of A. If $\operatorname{rad} A = \{0\}$, A is called **semisimple**.

Theorem 5.3

Let Δ be the maximal ideal space of a commutative Banach algebra A with unit.

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(b) *The Gelfand transform is a homomorphism of A onto a subalgebra $\hat{A}$ of $\mathcal{C}(\Delta)$, whose kernel is $\operatorname{rad} A$. The Gelfand transform is therefore an isomorphism if and only if A is semisimple.*

Theorem 5.3

Let Δ be the maximal ideal space of a commutative Banach algebra A with unit.

(a) *Δ is a compact Hausdorff space.*

(b) *The Gelfand transform is a homomorphism of A onto a subalgebra $\hat{A}$ of $\mathcal{C}(\Delta)$, whose kernel is* rad *A. The Gelfand transform is therefore an isomorphism if and only if A is semisimple.*

(c) *For each $x \in A$ we have $\operatorname{Rng} \hat{x} = \sigma(x)$.*

Theorem 5.4

If $\psi\colon B \to A$ is a homomorphism of a commutative Banach algebra B with unit into a semisimple commutative Banach algebra with unit, then ψ is continuous.

Lemma 5.5

If A is a commutative Banach algebra with unit and

$$r = \inf_{x \neq 0} \frac{\|x^2\|}{\|x\|^2}, \qquad s = \inf_{x \neq 0} \frac{\|\hat{x}\|_\infty}{\|x\|},$$

then $s^2 \leq r \leq s$.

5. Gelfand transformation

Theorem 5.6

Suppose A is a commutative Banach algebra with unit.

(a) *The Gelfand transform is an isometry if and only if* $\|x^2\| = \|x\|^2$.

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Suppose A is a commutative Banach algebra with unit.

(a) *The Gelfand transform is an isometry if and only if $\|x^2\| = \|x\|^2$.*

(b) *A is semisimple and $\hat{A}$ is closed in $\mathcal{C}(\Delta)$ if and only if there exists $K < \infty$ such that $\|x\|^2 \leq K\|x^2\|$ for every $x \in A$.*

5. Gelfand transformation

Definition

A mapping $x \mapsto x^*$ of a complex (not necessarily commutative) algebra A into A is called an **involution** on A if it has the following properties for every $x, y \in A$, and $\lambda \in \mathbf{C}$:

- $(x+y)^* = x^* + y^*$,
- $(\lambda x)^* = \overline{\lambda} x^*$,
- $(xy)^* = y^* x^*$,
- $x^{**} = x$.

Any $x \in A$ for which $x^* = x$ is called **hermitian**, or **self-adjoint**.

Theorem 5.7

If A is a Banach algebra with unit and an involution, and if $x \in A$, then

(a) $x + x^*$, $i(x - x^*)$ *and* xx^* *are hermitian,*

(b) x *has a unique representation* $x = u + iv$, *with* $u \in A$, $v \in A$, *and both* u *and* v *are hermitian,*

(c) *the unit* e *is hermitian,*

(d) x *is invertible in* A *if and only if* x^* *is invertible, in which case* $(x^*)^{-1} = (x^{-1})^*$, *and*

(e) $\lambda \in \sigma(x)$ *if and only if* $\overline{\lambda} \in \sigma(x^*)$.

5. Gelfand transformation

Theorem 5.8

If a Banach algebra A with unit is commutative and semisimple, then every involution on A is continuous.

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If a Banach algebra A with unit is commutative and semisimple, then every involution on A is continuous.

Definition
A Banach algebra A with an involution $x \mapsto x^*$ that satisfies $\|xx^*\| = \|x\|^2$ for every $x \in A$ is called a C^*-**algebra**.

5. Gelfand transformation

Theorem 5.9 (Gelfand-Naimark)

Suppose A is a commutative C^*-algebra with unit. The Gelfand transform is then an isometric isomorphism of A onto $\mathcal{C}(\Delta)$, which has the additional property $\widehat{x^*} = \overline{\hat{x}}$ for every $x \in A$.

5. Gelfand transformation

Theorem 5.10

If A is a commutative C^*-algebra with unit which contains an element x such that the polynomials in x and x^* are dense in A, then the formula $\widehat{\Psi f} = f \circ \hat{x}$ defines an isometric isomorphism Ψ of $\mathcal{C}(\sigma(x))$ onto A which satisfies $\Psi \overline{f} = (\Psi f)^*$ for every $f \in \mathcal{C}(\sigma(x))$. Moreover, if $f(\lambda) = \lambda$ on $\sigma(x)$, then $\Psi f = x$.

Definition

Let A be an algebra with an involution. If $x \in A$ and $xx^* = x^*x$, then x is said to be **normal**. A set $S \subset A$ is said to be **normal** if S commutes and if $x^* \in S$ whenever $x \in S$.

Theorem 5.11

Suppose A is a Banach algebra with an involution, and B is a normal subset of A that is maximal with respect to being normal. Then

(a) *B is a closed commutative subalgebra of A, and*

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(a) *B is a closed commutative subalgebra of A, and*

(b) *$\sigma_B(x) = \sigma_A(x)$ for every $x \in B$.*

Theorem 5.12

Every C^*-algebra A has the following properties:

(a) *Hermitian elements have real spectra.*

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(d) *If $u, v \in A$ are hermitian, $\sigma(u) \subset [0, \infty)$, $\sigma(v) \subset [0, \infty)$, then $\sigma(u + v) \subset [0, \infty)$.*

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(e) *If $y \in A$, then $\sigma(yy^*) \subset [0, \infty)$.*

Theorem 5.13

Suppose that A is a C^*-algebra with a unit e, B is a closed subalgebra of A, $e \in B$, and $x^* \in B$ for every $x \in B$. Then $\sigma_A(x) = \sigma_B(x)$ for every $x \in B$.

6. Operators on Hilbert spaces

In this section the symbol H stands for a nontrivial complex Hilbert space.

Definition

We say that $T \in \mathcal{L}(H)$ is

- **normal**, if $T^*T = TT^*$,
- **selfadjoint** (or also **hermitian**), if $T^* = T$,
- **unitary**, if $T^*T = I = TT^*$,
- **orthogonal projection**, if T is a projection, i.e., $T = T^2$, and $\text{Rng}\, T \perp \text{Ker}\, T$.

Lemma 6.1

Let $T \in \mathcal{L}(H)$*. Then*

(a) $\|T^*T\| = \|TT^*\| = \|T\|^2$,

(b) $\text{Ker } T^* = \text{Rng } T^\perp$.

Lemma 6.1

Let $T \in \mathcal{L}(H)$. *Then*

(a) $\|T^*T\| = \|TT^*\| = \|T\|^2$,

(b) $\text{Ker } T^* = \text{Rng } T^{\perp}$.

Lemma 6.2

Let $T \in \mathcal{L}(H)$. *Then the following are equivalent*

(i) $T = 0$,

(ii) $(Tx, x) = 0$ *for every* $x \in H$.

Corollary 6.3

Let $S, T \in \mathcal{L}(X)$ *for every* $x \in H$ *satisfy* $(Sx, x) = (Tx, x)$. *Then* $T = S$.

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Theorem 6.4 (characterization of normal operators)

An operator $T \in \mathcal{L}(H)$ *is normal if and only if* $\|Tx\| = \|T^*x\|$ *for each* $x \in H$.

Theorem 6.5 (properties of normal operators)

Let $T \in \mathcal{L}(H)$ be normal. Then we have

(a) $\text{Ker } T = \text{Ker } T^*$,

(b) *T is invertible if and only if* **bounded from below**, *i.e., there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for every $x \in H$ (Weyl),*

(c) *if $x \in H$ satisfies $Tx = \lambda x$, then $T^*x = \overline{\lambda}x$,*

(d) *if $\lambda_1, \lambda_2 \in \mathbf{C}$ are different eigenvalues of T, then* $\text{Ker}(\lambda_1 I - T) \perp \text{Ker}(\lambda_2 I - T)$,

(e) $\|T^2\| = \|T\|^2$,

(f) $\|T\| = \rho(T)$.

Theorem 6.6 (characterization of selfadjoint operators)

Let $T \in \mathcal{L}(H)$. Then $T = T^*$ if and only if (Tx, x) is a real number for every $x \in H$.

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Theorem 6.8

For every $T \in \mathcal{L}(H)$ there exists a unique decomposition $T = S_1 + iS_2$, where S_1, S_2 are selfadjoint operators.

6. Operators on Hilbert spaces

Definition

Let $T \in \mathcal{L}(H)$. **Numerical range** of the operator T is defined by

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Theorem 6.9 (Hilbert–Toeplitz)

Let $T \in \mathcal{L}(H)$. Then $\sigma(T) \subset \overline{N(T)}$.

Theorem 6.10 (spectrum of selfadjoint operator)

Let $T \in \mathcal{L}(H)$ be selfadjoint. Then $N(T) \subset \mathbf{R}$ and if we denote $m_T = \inf N(T)$, $M_T = \sup N(T)$, then we have

(i) $\sigma(T) \subset [m_T, M_T]$,

(ii) $\|T\|$ *or* $-\|T\|$ *is in* $\sigma(T)$,

(iii) $m_T, M_T \in \sigma(T)$.

Theorem 6.11 (characterization of unitary operators)

Let $U \in \mathcal{L}(H)$. Then the following are equivalent:

(i) *U is unitary,*

(ii) $\operatorname{Rng} U = H$ *a* $(Ux, Uy) = (x, y)$, $x, y \in H$,

(iii) $\operatorname{Rng} U = H$ *a* $\|Ux\| = \|x\|$, $x \in H$.

Theorem 6.12 (characterization of orthogonal projections)

Let $P \in \mathcal{L}(H)$ be a projection. Then the following are equivalent:

(i) *P is selfadjoint,*

(ii) *P is normal,*

(iii) *P is orthogonal,*

(iv) $(Px, x) = \|Px\|^2$, $x \in H$.

Theorem 6.13 (spectral decomposition of compact normal operator; Hilbert–Schmidt)

Let $T \in \mathcal{L}(H)$ be compact and normal. Then there exists an orthonormal basis of H formed by eigenvectors of T. Further there exist nonzero eigenvalues $\{\lambda_n\}_{n=1}^m$, $m \in \mathbf{N} \cup \{\infty\}$, and an orthonormal basis $\{e_n\}_{n=1}^m$ of the space $\overline{\operatorname{Rng} T}$ such that

$$Tx = \sum_{n=1}^{m} \lambda_n (x, e_n) e_n, \quad x \in H.$$

7. Spectral decomposition

Theorem 7.1

Let $T \in \mathcal{L}(H)$ be normal. Then there exists a calculus $\Psi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(H)$ with the following properties:

(1) $\Psi(p) = \sum_{k,l=0}^{n} a_{kl} T^k (T^*)^l$ *for* $p(z) = \sum_{k,l=0}^{n} a_{kl} z^k \overline{z}^l$,

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(7) *if* S *commutes with* T, *then* S *commutes with* $\Psi(f)$.

Lemma 7.2 (Lax-Milgram)

Let $B\colon H \times H \to \mathbf{C}$ be linear in the first coordinate and conjugate linear in the second coordinate. Let

$$M := \sup_{x,y \in B_H} |B(x,y)| < \infty$$

Then there exists a unique $T \in \mathcal{L}(H)$ with $B(x,y) = (Tx, y)$ for $x, y \in H$ and $\|T\| = M$.

7. Spectral decomposition

Notation

Let P be a metric space, then $\mathcal{B}^b(P)$ denotes the set of all bounded Borel functions from P to $\mathbf{C}$. The set $\mathcal{B}^b(P)$ is equipped by the supremum norm.

7. Spectral decomposition

Notation

Let P be a metric space, then $\mathcal{B}^b(P)$ denotes the set of all bounded Borel functions from P to $\mathbf{C}$. The set $\mathcal{B}^b(P)$ is equipped by the supremum norm.

Lemma 7.3

Let P be a compact metric space and $\mathcal{A}$ be the smallest system of complex function on P, which contains continuous functions and is closed with respect to pointwise limit of bounded sequences. Then $\mathcal{A} = \mathcal{B}^b(P)$.

Theorem 7.4

Let $T \in \mathcal{L}(H)$ *be normal. Then there exists a Borel calculus* $\Theta\colon \mathcal{B}^b(\sigma(T)) \to \mathcal{L}(H)$ *such that*

(1) $\Theta = \Psi$ *on* $\mathcal{C}(\sigma(T))$,

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7. Spectral decomposition

Notation

Let K be a metric space. The system of all Borel subsets of K is denoted by $\mathrm{Borel}(K)$.

7. Spectral decomposition

Definition

Let K be a nonempty compact metric space. We say that the mapping $E\colon \operatorname{Borel}(K) \to \mathcal{L}(H)$ is **spectral measure**, if we have:

(i) for every $B \in \operatorname{Borel}(K)$ is $E(B)$ an orthogonal projection, $E(\emptyset) = 0$, $E(K) = I$,

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(iv) for every $x \in H$ the mapping $E_{x,x}\colon B \mapsto (E(B)x, x)$ is a measure on K, such that its completion is Radon.

7. Spectral decomposition

Theorem 7.5

If $T \in \mathcal{L}(H)$ *is normal, then* $E\colon \operatorname{Borel}(\sigma(T)) \to \mathcal{L}(H)$ *defined as* $E(B) = \Theta(\chi_B)$ *is a spectral measure and it holds:*

(i) $\forall x \in H \; \forall f \in \mathcal{B}^b(\sigma(T))\colon \; (\Theta(f)x, x) = \int_{\sigma(T)} f \,\mathrm{d}E_{x,x}$,

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If $T \in \mathcal{L}(H)$ is normal, then $E\colon \operatorname{Borel}(\sigma(T)) \to \mathcal{L}(H)$ defined as $E(B) = \Theta(\chi_B)$ is a spectral measure and it holds:

(i) $\forall x \in H\ \forall f \in \mathcal{B}^b(\sigma(T)):\ (\Theta(f)x, x) = \int_{\sigma(T)} f \,\mathrm{d}E_{x,x}$,

(ii) *for $A \in \operatorname{Borel}(\sigma(T))$ and $T_A := T|_{\operatorname{Rng} E(A)}$ we have $T_A \in \mathcal{L}(\operatorname{Rng} E(A))$ and $\sigma(T_A) \subset \overline{A}$,*

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(iii) *for every nonempty set $G \subset \sigma(T)$ which is open in $\sigma(T)$ we have $E(G) \neq 0$.*

Theorem 7.6

Let $E\colon \operatorname{Borel}(K) \to \mathcal{L}(H)$ *be a spectral measure on a nonempty compact metric space* K. *For every function* $f \in \mathcal{B}^b(K)$ *there exists a unique* $T(f) \in \mathcal{L}(H)$ *satisfying* $(T(f)x, x) = \int_K f \,\mathrm{d}E_{x,x}$ *for every* $x \in H$. *Further we have*

(i) *the mapping* $T\colon f \mapsto T(f)$ *is linear, multiplicative,* $\|T\| = 1$, *and* $T\big(\overline{f}\big) = \big(T(f)\big)^*$,

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Notation

We denote $T(f) = \int_K f \, \mathrm{d}E = \int_K f(t) \, \mathrm{d}E(t)$.

7. Spectral decomposition

Theorem 7.7

Let $T \in \mathcal{L}(H)$ be normal. Then there exists a unique spectral measure E on $\sigma(T)$ such that $T = \int_{\sigma(T)} t \, \mathrm{d}E(t)$.

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Let $T \in \mathcal{L}(H)$ be normal and $\lambda \in \sigma(T)$. Then we have

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(i) $\operatorname{Rng} E(\{\lambda\}) = \operatorname{Ker}(\lambda I - T)$,

(ii) $\lambda \in \sigma_p(T)$ *if and only if* $E(\{\lambda\}) \neq 0$,

(iii) *if* λ *is an isolated point of* $\sigma(T)$, *then* $\lambda \in \sigma_p(T)$.

Definition

We say that $T \in \mathcal{L}(H)$ is **positive** if $(Tx, x) \geq 0$ for every $x \in H$. If T is positive we write $T \leq 0$.

Theorem 7.9

Let $T \in \mathcal{L}(H)$. Then the following are equivalent

(i) $\forall x \in H\colon (Tx, x) \geq 0$,

(ii) $T = T^*$ *and* $\sigma(T) \subset [0, \infty)$.

7. Spectral decomposition

Theorem 7.10

Every positive $T \in \mathcal{L}(H)$ *has a unique positive square root* $S \in \mathcal{L}(H)$. *If* T *is invertible then* S *is invertible.*

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If $T \in \mathcal{L}(H)$, *then the positive square root of* T^*T *is the only positive operator* $P \in \mathcal{L}(H)$ *that satisfies* $\|Px\| = \|Tx\|$ *for every* $x \in H$.

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(a) *If $T \in \mathcal{L}(H)$ is invertible, then T has a unique* **polar decomposition** *$T = UP$, i.e., U is unitary and $P \geq 0$.*

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(a) *If* $T \in \mathcal{L}(H)$ *is invertible, then* T *has a unique* **polar decomposition** $T = UP$*, i.e.,* U *is unitary and* $P \geq 0$*.*

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