Mathematics I

Entrance test, 3. 10. 2018

1. Find all real solutions of the equation $2x^2 + 3x - 2 = 0$.

2. Find the set M of all solutions of the equation $e^{x^2-8x+7} \leq 1$ in the real domain.

3. Compute the distance d of the point $[1, 0]$ from the line which contains the points $[1, -1]$ and $[-2, 2]$.

4. For every real numbers a, b the term $\sqrt{a^2 + 4ab + 4b^2}$ equals

a) $\sqrt{(a + 2b)^2}$,
b) $a + 2b$,
c) $|a| + 2\,|b|$,
d) $|a + 2b|$,
e) $\big||a| + 2\,|b|\big|$.

5. Solve $\frac{x-1}{2x+16} > 1$ in the real domain.

6. Solve the equation $3\cos^2 x = 3\cos x - \sin^2 x$ in the real domain.

7. Solve the system of two equations with a real parameter λ.

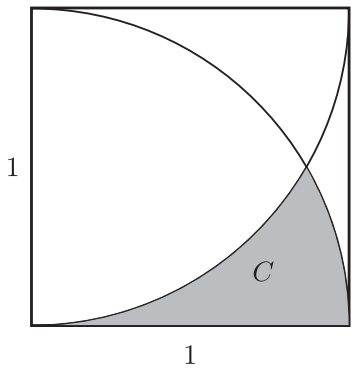
$$\lambda x + y = 1$$
$$x + \lambda y = 1$$

8. Draw the graph of the function: $f(x) = \big||x + 1| - 2\big|$. Let a be a real number. Consider the equation $f(x) = a$. Mark all correct answers.

a) There exists $a \in \mathbb{R}$ such that the equation has exactly three solution.
b) There exists $a \in \mathbb{R}$ such that the equation has exactly one solution.
c) For each $a \in \mathbb{R}$ each solution is contained in the interval $\langle -a, a\rangle$.
d) The equation has a solution for each $a \in \mathbb{R}$.
e) The function f is even.

9. Find all lines which are tangent to the circles $\{[x, y] \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$ and $\{[x, y] \in \mathbb{R}^2;\ (x - 4)^2 + y^2 = 1\}$.

10. In the square with the side of the length 1 consider the set C whose boundary is formed by one side of the square and by parts of the circles centered at the vertices of the square with radius 1 (see the picture). Find the area S of the set C.



Answers

1. The roots are $-2, \frac{1}{2}$.

2. $M = \langle 1, 7 \rangle$

3. The line is determined by the equation $y = -x$. The distance of $[1, 0]$ from this line is $\frac{1}{\sqrt{2}}$.

4. The term equals $\sqrt{(a+2b)^2} = |a+2b|$.
Correct answers: a, d.

5. $M = (-17, -8)$

6. Substituting $y = \cos x$ gives the equation $2y^2 - 3y + 1 = 0$ having the roots $\frac{1}{2}$ a 1. Solution: $\frac{\pi}{3} + 2k\pi, -\frac{\pi}{3} + 2k\pi, 2k\pi$, where $k \in \mathbb{Z}$.

7. The system has for $\lambda = 1$ infinitely many solutions, for $\lambda = -1$ there is no solution and for $\lambda \neq \pm 1$ here is a unique solution $\left[\frac{1}{1+\lambda}, \frac{1}{1+\lambda}\right]$.

8. Correct answers: a.

9. There are four solutions: $y = -1$, $y = 1$, $y = \frac{1}{\sqrt{3}}(x-2)$, $y = -\frac{1}{\sqrt{3}}(x-2)$.

10. $S = \frac{\sqrt{3}}{4} - \frac{\pi}{12}$