INFINITE GAMES AND σ-POROSITY

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Abstract. We show a game characterizing various types of σ-porosity in terms of winning strategies. We use the game to prove and reprove some new and older inscribing theorems for σ-ideals of σ-porous type in locally compact metric spaces.

1. Introduction

The theory of porous and σ-porous sets forms an important part of real analysis and Banach space theory for more than forty years. It originated in 1967 when E. P. Dolženko used for the first time the nomenclature 'porous set' and proved that some sets of his interest are σ-porous ([2]). Since then the porosity has been used many times as well as many variants of this notion (see Section 4). The interested reader can consult the survey papers of L. Zajíček ([10, 12]) on porous and σ-porous sets.

Here we are interested in structural properties of σ-ideals of σ-porous type. More precisely, the main question we will consider in this work is the following one.

Question. *Let A be an analytic subset of a metric space X and $\mathcal{I}$ be a σ-ideal of subsets of X. Suppose that $A \notin \mathcal{I}$. Does there exist a closed set $F \subset A$ which is not in $\mathcal{I}$?*

This question was posed by L. Zajíček in [10] (for a Borel set A) for classical Dolženko σ-porosity. An affirmative answer was given independently by J. Pelant (for any topologically complete metric space X) and M. Zelený (for any compact metric space X). Their results are demonstrated in a joint paper [13] which combines the original idea of J. Pelant (giving an explicit construction of the set F) and techniques developed by M. Zelený. The case of some other types of porosity (including the ordinary one in a locally compact metric space X but also $\langle g\rangle$-porosity in a locally compact metric space X and symmetrical porosity in $\mathbb{R}$) was solved (also affirmatively) by M. Zelený and L. Zajíček in [14]. They offer a less complicated method of construction of F using so called 'porosity-like' relations. Their non-constructive proof uses tools from Descriptive Set Theory. However, the authors admitted that their method cannot be applied to strong porosity and so Question for strong porosity still remained open (even in a compact metric space X).

Later on J. Zapletal introduced a new powerful tool to describe σ-porous sets. This was an infinite game which can be used to characterize σ-porous sets in $2^{\mathbb{N}}$ considered with respect to certain metric compatible with the product topology. This game is used to

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reprove the positive answer to Question in this particular case ([5, Example 4.20]). The only attempt to answer Question for strong porosity (and ordinary porosity once again) was made by D. Rojas-Rebolledo, who generalized in [9] the ideas from [5]. He managed to give an affirmative answer to Question in any zero-dimensional compact metric space X. Further, M. Doležal ([1]) showed a characterization of σ-P-porous sets for any porosity-like relation P via an infinite game.

Our aim is to generalize results of [5, 9, 14] in two directions. We give a positive answer to Question in spaces which are more general than those considered in [5, 9] and also for σ-ideals of σ-porous type which are not included in [14].

Let us look at the contents of this work a little closer. Definitions and well known results necessary in the sequel are presented in Section 2. In Section 3, we prove the main result of the paper (Theorem 3.11). The complete formulation is a little bit technical so let us formulate the result in an informal way.

Let X be a compact metric space and R be a porosity-like relation on X satisfying some additional conditions. Then every analytic non-σ-R-porous subset A of X contains a compact non-σ-R-porous subset.

To prove this we proceed as follows. We introduce a variant of Zapletal's game for two players played with a set A. We prove that the second player has a winning strategy if and only if A is σ-R-porous. Now consider non-σ-R-porous analytic subset A of X. By the result of Farah and Zapletal ([5, Theorem 4.16]) we may assume that A is non-σ-R-porous and *Borel*. Then we show that our game with A is determined using Martin Determinacy Theorem. The set A is non-σ-R-porous thus the second player does not have a winning strategy. By determinacy the first player has to have a winning strategy. Using a winning strategy of the first player we find a compact subset K of A such that the first player still has a winning strategy in the game played with K. This means that the second player does not have a winning strategy and so K is not σ-R-porous.

In Section 4, we apply the last result to concrete porosities and obtain an (affirmative) answer to several different variants of Question. Namely, we deal with ordinary porosity, strong porosity, strong right porosity, and 1-symmetrical porosity. As it is described earlier, the first result have been already known but the method used in our work (based on an infinite game) aspires to be more elegant and easier than the known proofs. The other results are new. Finally, we show that there exists a closed set in $\mathbb{R}$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $\varepsilon \in (0,1)$ but which is not σ-1-symmetrically porous. This answers a question posed by M. J. Evans and P. D. Humke in [4].

2. Preliminaries

Let (X,d) be a metric space. An open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$. Since an open ball (considered as a set) does not uniquely determine its center and radius, we will identify every open ball with the pair (center, radius) throughout this work. Therefore two different open balls (i.e., two different pairs (center, radius)) can still determine the same subset of X. Now, for $p > 0$ and an open ball B with center $x \in X$

and radius $r > 0$, we can define $p \star B$ as an open ball with center x and radius pr. The closed ball with center $x \in X$ and radius $r > 0$ is denoted by $\overline{B}(x, r)$. We employ the same identification of closed balls with the pair (center, radius) as for open balls. If $A \subset X$ is nonempty and $r > 0$ then $B(A, r) = \{x \in X\colon \operatorname{dist}(x, A) < r\}$. We also set $B(\emptyset, r) = \emptyset$.

We will prove our results for porosity-like relations satisfying some additional assumptions and then apply it to concrete cases. To do this, we need the following definition.

Definition 2.1. Let X be a metric space and let $P \subset X \times 2^X$ be a relation between points of X and subsets of X. Then P is called a *point-set relation on* X. The symbol $P(x, A)$ where $x \in X$ and $A \subset X$ means that $(x, A) \in P$. A point-set relation P on X is called a *porosity-like relation* if the following conditions hold for every $A \subset X$ and $x \in X$:

(P1) if $B \subset A$ and $P(x, A)$ then $P(x, B)$,
(P2) we have $P(x, A)$ if and only if there exists $r > 0$ such that $P(x, A \cap B(x, r))$,
(P3) we have $P(x, A)$ if and only if $P(x, \overline{A})$.

If P is a porosity-like relation on X, $A \subset X$, and $x \in X$, we say that

- A is *P-porous at x* if $P(x, A)$,
- A is *P-porous* if it is P-porous at each $x \in A$,
- A is *σ-P-porous* if it is a countable union of P-porous sets.

We will need the following theorem.

Theorem 2.2 ([11, Lemma 3])**.** *Let X be a metric space, P be a porosity-like relation on X, and $A \subset X$. Then A is σ-P-porous if and only if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap A$ is σ-P-porous.*

Further let us recall that if s is a finite sequence of elements of a nonempty set A and t is a finite or infinite sequence of elements of A, then the *concatenation* of s and t is denoted by $s^\wedge t$.

3. Main result

3.1. **The class $\mathfrak{R}$.** Now we define the class of porosity-like relations for which we prove positive answer to our Question in compact metric spaces. The definition is technical but it covers many interesting concrete cases and verification of the conditions is straightforward.

Definition 3.1. Let (X, d) be a nonempty metric space. We say that a point-set relation R on X belongs to the class $\mathfrak{R}(X)$ if there are point-set relations R^s and $R_r^{s,q}$ on X, $s \in \mathbb{N}$, $r > 0$, $q \in (0, 1)$, such that the following conditions are satisfied:

(R1) $R^s = \bigcap\limits_{0<q<1} \bigcap\limits_{R>0} \bigcup\limits_{0<r<R} R_r^{s,q}$ and $R = \bigcup\limits_{s\in\mathbb{N}} R^s$,
(R2) if $R_r^{s,q}(x, A)$ and $0 < w < \frac{q}{2s}$ then $R_r^{s,q-2sw}(x, B(A, rw))$,
(R3) if $B \subset A$ and $R_r^{s,q}(x, A)$ then $R_r^{s,q}(x, B)$,
(R4) we have $R_r^{s,q}(x, A)$ if and only if $R_r^{s,q}(x, A \cap B(x, 2r))$,
(R5) the set $\{(x, r) \in X \times (0, \infty)\colon R_r^{s,q}(x, A)\}$ is open in $X \times (0, \infty)$.

Convention 3.2. Throughout this section we will work with a fixed compact metric space K with a fixed point-set relation $R \in \mathfrak{R}(K)$. The corresponding point-set relations $R_r^{s,q}, R^s$ witnessing $R \in \mathfrak{R}(K)$ are fixed as well. We also fix a set $A \subset K$.

Lemma 3.3. *Let $s \in \mathbb{N}$ and $r > 0$. Then we have:*

(M) *if $0 < q_1 < q_2 < 1$ and $R_r^{s,q_2}(x, A)$, then $R_r^{s,q_1}(x, \overline{A})$,*

(P) *R^s is a porosity-like relation; consequently, R is a porosity like-relation.*

Proof. (M) By (R2) applied to $w = \frac{q_2 - q_1}{2s}$, we get $R_r^{s,q_1}\big(x, B(A, \frac{r(q_2-q_1)}{2s})\big)$. By (R3), we have $R_r^{s,q_1}(x, \overline{A})$.

We verify (P1)–(P3) from Definition 2.1 to prove (P).

(P1) This property is an immediate consequence of (R1) and (R3)

(P2) Suppose that $R^s(x, A \cap B(x, r_0))$ for some $r_0 > 0$. By (R1), there exist sequences $(q_k)_{k=1}^\infty$ of real numbers from $(0,1)$ and $(r_k)_{k=1}^\infty$ of real numbers from $(0,\infty)$ such that $\lim_{k\to\infty} q_k = 1$, $\lim_{k\to\infty} r_k = 0$, and $R_{r_k}^{s,q_k}(x, A \cap B(x, r_0))$ for every $k \in \mathbb{N}$. There exists $k_0 \in \mathbb{N}$ such that $2r_k \le r_0$ for every $k \ge k_0$. Then $R_{r_k}^{s,q_k}(x, A \cap B(x, 2r_k))$ for $k \ge k_0$ by (R3) and so $R_{r_k}^{s,q_k}(x, A)$ for $k \ge k_0$ by (R4). Using (R1) and (M), we get $R^s(x, A)$. The opposite implication follows by (P1).

(P3) Suppose that $R^s(x, A)$. Choose $q \in (0,1)$ and $R > 0$. By (R1), there exists $0 < \tilde{r} < R$ such that $R_{\tilde{r}}^{s,q}(x, A)$. By (M) we have $R_{\tilde{r}}^{s,q'}(x, \overline{A})$ for every $0 < q' < q$. Using (R1) we get $R^s(x, \overline{A})$. The opposite implication follows by (P1).

The fact that R is a porosity-like relation follows directly from the definition of R. $\square$

3.2. **Boulder-Sisyfos game.** For the rest of this section, let us fix sequences $(R_n)_{n=1}^\infty$ and $(a_n)_{n=1}^\infty$ of real numbers from $(0,\infty)$ such that for every $n \in \mathbb{N}$

$$R_{n+1} \le 2^{-(n+2)} R_n \tag{1}$$

and

$$\lim_{n\to\infty} \frac{a_n}{R_{n+2}} = 0. \tag{2}$$

Let D_n, $n \in \mathbb{N}$, be a finite a_n-net in K (i.e., a finite subset of K such that $K = \bigcup\{B(y, a_n) \colon y \in D_n\}$) and let $M_n = \{B(y, a_n) \colon y \in D_n\}$.

Let A be an arbitrary subset of K. We define a game $H(A)$ for two players, who will be called Boulder and Sisyfos. These names were used by J. Zapletal in the original version of his game. The game is played as follows:

Boulder	B_1		B_2		B_3		$\dots$
Sisyfos		(S_1^1)		(S_2^1, S_2^2)		(S_3^1, S_3^2, S_3^3)	

On the first move, Boulder plays an open ball $B_1 \subset K$ with radius R_1 and Sisyfos plays an open set $S_1^1 \subset B_1$ where S_1^1 is a union (possibly empty) of some balls from M_1. On the second move, Boulder plays an open ball B_2 with center in $\frac{1}{2}\star B_1$ and radius R_2 and Sisyfos plays two open sets S_2^1 and S_2^2 such that $S_2^1 \cup S_2^2 \subset B_2$ where S_2^j is a union of some balls from M_2, $j = 1, 2$. On the nth move, $n > 1$, Boulder plays an open ball B_n with center in $(1 - 2^{-(n-1)}) \star B_{n-1}$ and radius R_n and Sisyfos replies by playing open sets $S_n^1, S_n^2, \dots, S_n^n$ such that $\bigcup_{j=1}^n S_n^j \subset B_n$ where S_n^j is a union of some balls from M_n, $j = 1, 2, \dots, n$.

By (1), we have $\lim_{n\to\infty} \operatorname{diam} B_n = 0$. Using this fact and the compactness of K, when a run of the game is over, we get a unique point x lying in the intersection of the balls B_n, $n \in \mathbb{N}$, played by Boulder. We call this point an *outcome* of the run. Sisyfos wins if at least one of the following conditions is satisfied:

(a) $x \notin A$,

(b) there exists $m \in \mathbb{N}$ such that one can find $s \in \mathbb{N}$, sequences $(n_k)_{k=1}^\infty$ of integers from $\{m, m+1, \dots\}$, $(q_k)_{k=1}^\infty$ of real numbers from $(0, 1)$, and $(r_k)_{k=1}^\infty$ of real numbers from $(0, \infty)$ such that
- $x \in K \setminus \bigcup\limits_{n=m}^{\infty} S_n^m$,
- $\lim\limits_{k\to\infty} n_k = \infty$,
- $\lim\limits_{k\to\infty} q_k = 1$,
- $r_k \le 2^{-(n_k+3)} R_{n_k}$, $k \in \mathbb{N}$,
- $R_{r_k}^{s,q_k}(x, K \setminus S_{n_k}^m)$, $k \in \mathbb{N}$.

Boulder wins in the opposite case. If condition (b) is satisfied for some $m \in \mathbb{N}$, then m is called a *witness of Sisyfos' victory.*

At first sight, condition (b) looks very complicated. For a better understanding, we can observe that it is stronger than the assertion that $R^s(x, K \setminus \bigcup_{n=m}^\infty S_n^m)$ by (R1), (R3), and (M).

We use the above notation in the next lemma.

Lemma 3.4. *For every $n \in \mathbb{N}$, we have $B_{n+1} \subset \left(1 - \frac{1}{2^{n+1}}\right) \star B_n$.*

Proof. Suppose that x_n is the center of B_n, x_{n+1} is the center of B_{n+1}, and $z \in B_{n+1}$. Then we have

$$\begin{aligned} d(z, x_n) &\le d(z, x_{n+1}) + d(x_{n+1}, x_n) < R_{n+1} + \left(1 - 2^{-n}\right) R_n \\ &\le \left(2^{-(n+2)} + 1 - 2^{-n}\right) R_n = \left(1 - 3 \cdot 2^{-(n+2)}\right) R_n < \left(1 - 2^{-(n+1)}\right) R_n. \end{aligned}$$

□

3.3. **Characterization of σ-R-porosity via the infinite game.** In this subsection we show that the notion of σ-R-porosity, where R is the fixed porosity-like relation belonging to the class $\mathfrak{R}$, can be characterized by existence of a winning strategy of Sisysfos in our game. To this end we will need couple of auxiliary notions.

We say that a finite (also empty) sequence of open balls $(B_1, B_2, \dots, B_i)$ is *good* if B_{n+1} is centered at $\frac{1}{2} \star B_n$, $n = 1, \dots, i-1$, and the radius of B_n equals R_n, $n = 1, \dots, i$. That is, a finite sequence of open balls is good if the rules of the game $H(A)$ allow Boulder to play the ball B_n on his nth move, $n = 1, 2, \dots, i$.

For $n, m \in \mathbb{N}$ we define

$$d_n^m = \begin{cases} 1 - 2^{-n+m-1} & \text{if } m \leq n, \\ \frac{1}{4} & \text{if } m > n. \end{cases}$$

Let σ be a strategy for Sisyfos in the game $H(A)$. If $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$ then we say that a good sequence of open balls $(B_1, B_2, \ldots, B_i)$ is (k,l)-*good* (with respect to the strategy σ) if there exists a run of the game $H(A)$ such that the following conditions hold:

- Sisyfos followed the strategy σ,
- Boulder played the ball B_n on his nth move, $n = 1, 2, \ldots, i$,
- the following conditions are satisfied for every positive $n \in \{k, k+1, \ldots, i-1\}$:
  - (H1) if $\left[l > n \text{ or } \left(l \leq n \text{ and } S_n^l \cap \left(d_n^l \star B_n\right) = \emptyset\right)\right]$ then the center of B_{n+1} lies in $d_n^{l+1} \star B_n$,
  - (H2) if $\left[l \leq n \text{ and } S_n^l \cap \left(d_n^l \star B_n\right) \neq \emptyset\right]$ then the center of B_{n+1} lies in $d_n^l \star B_n$.

Let Boulder and Sisyfos play a run of the game $H(A)$. Let $V = (B_1, \mathcal{S}_1, B_2, \mathcal{S}_2, \ldots)$, and $\mathcal{S}_n = (S_n^1, S_n^2, \ldots, S_n^n)$, $n \in \mathbb{N}$, where Boulder played the ball B_n and Sisyfos played the sets $S_n^1, S_n^2, \ldots, S_n^n$ on the nth move of the run. Then we will refer to the run itself by V and if we talk about the ball B_n or about the set S_n^m, we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

We say that a run V of the game $H(A)$ is (k,l)-*good* if Sisyfos followed the strategy σ and the sequence $(B_1(V), B_2(V), \ldots, B_j(V))$ is (k,l)-good for every $j \in \mathbb{N}$.

It is easy to see that if $l_1 > l_2$ and a finite sequence of open balls (a run of the game $H(A)$, respectively) is (k, l_1)-good then it is also (k, l_2)-good.

If $T = (B_1, B_2, \ldots, B_i)$ is a good sequence of open balls, we say that a run V of the game $H(A)$ is T-*compatible* if $B_n(V) = B_n$ for every $n \in \{1, 2, \ldots, i\}$.

For $m \in \mathbb{N} \cup \{0\}$ and a good sequence of open balls $T = (B_1, B_2, \ldots, B_i)$, we denote by $M_m(T)$ the set of all

$$x \in \begin{cases} A & \text{if } T = \emptyset, \text{ i.e., } i = 0, \\ A \cap \left(\frac{1}{4} \star B_i\right) & \text{if } i > 0 \end{cases}$$

such that in every T-compatible $(i, m+1)$-good run of the game $H(A)$ giving x as its outcome, all the witnesses of Sisyfos' victory (if there exist any) are greater than m. The set $M_m(T)$ also depends on the set A and on the strategy σ but these will be always fixed.

Lemma 3.5. *Let σ be a strategy for Sisyfos in the game $H(A)$. Let $T_0 = (B_1, B_2, \ldots, B_i)$ be a good sequence of open balls and $m \in \mathbb{N} \cup \{0\}$. Then there exist an R-porous set $N_m(T_0)$ and an at most countable collection $\mathcal{T}$ of finite sequences of open balls such that $T_0{}^\wedge T$ is $(i, m+1)$-good for every $T \in \mathcal{T}$ and*

$$M_m(T_0) \subset N_m(T_0) \cup \bigcup \{M_{m+1}(T_0{}^\wedge T) \colon T \in \mathcal{T}\}.$$

Proof. Define $N_m(T_0)$ as the set of all $x \in M_m(T_0)$ such that

(I) there exists a T_0-compatible $(i, m+2)$-good run of the game $H(A)$ giving x as its outcome such that $m+1$ is a witness of Sisyfos' victory,

(II) for every T_0-compatible $(i, m+2)$-good run V of the game $H(A)$ and for every $n \geq \max\{i, m+1\}$, we have $x \notin S_n^{m+1}(V) \cap (d_n^{m+1} \star B_n(V))$.

Suppose that $x \in M_m(T_0) \setminus \big(M_{m+1}(T_0) \cup N_m(T_0)\big)$. By definition of $M_{m+1}(T_0)$ there exists a T_0-compatible $(i, m+2)$-good run with the outcome x and with a witness less or equal $m+1$. Since $x \in M_m(T_0)$ and the run is also $(i, m+1)$-good, the witness is equal $m+1$. Thus condition (I) holds for x. Therefore condition (II) cannot be true by the definition of $N_m(T_0)$, and so there exist a T_0-compatible $(i, m+2)$-good run $V(x)$ of the game $H\,(A)$ and $n(x) \geq \max\{i, m+1\}$ such that

$$x \in S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}\big(V(x)\big)\right).$$

Denote $B_j(x) = B\,(x, R_j)$ for $j > n(x)$. Find $N(x) > n(x)$ such that $B_{N(x)}(x) \subset S_{n(x)}^{m+1}(V(x))$ and denote

$$T(x) = \big(B_{i+1}(V(x)), \ldots, B_{n(x)}(V(x))\big) {}^\wedge \left(B_{n(x)+1}(x), \ldots, B_{N(x)}(x)\right).$$

Then the sequence $T_0{}^\wedge T(x)$ is $(i, m+1)$-good. Indeed, the sequence

$$T_0{}^\wedge \left(B_{i+1}(V(x)), \ldots, B_{n(x)}(V(x))\right)$$

is even $(i, m+2)$-good and the fact that

$$S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}\big(V(x)\big)\right) \neq \emptyset$$

allows Boulder to use condition (H2) and play the ball with center $x \in d_{n(x)}^{m+1} \star B_{n(x)}(V(x))$ on his $(n\,(x)+1)$st move. Since $B_{N(x)}(x) \subset S_{n(x)}^{m+1}(V(x))$, we see that $m+1$ cannot become a witness of Sisyfos' victory in any $T_0{}^\wedge T(x)$-compatible run of the game $H\,(A)$. Therefore we have

$$M_m(T_0) \cap \left(\tfrac{1}{4} \star B_{N(x)}(x)\right) \subset M_{m+1}(T_0{}^\wedge T(x)),$$

and so $x \in M_{m+1}(T_0{}^\wedge T(x))$. By Lindelöf's property, there exists an at most countable set

$$\{x_j \colon j \in \mathbb{N}\} \subset M_m(T_0) \setminus (M_{m+1}(T_0) \cup N_m(T_0))$$

such that $M_m(T_0) \setminus \big(M_{m+1}(T_0) \cup N_m(T_0)\big)$ is covered by the system $\left\{\frac{1}{4} \star B_{N(x_j)}(x_j) \colon j \in \mathbb{N}\right\}$ of open sets and so it is also covered by the countable system $\{M_{m+1}(T_0{}^\wedge T(x_j)) \colon j \in \mathbb{N}\}$. Now, we can define $\mathcal{T} = \{\emptyset\} \cup \{T(x_j) \colon j \in \mathbb{N}\}$. Then we obviously have

$$M_m(T_0) \subset N_m(T_0) \cup \bigcup \{M_{m+1}(T_0{}^\wedge T) \colon T \in \mathcal{T}\}.$$

It remains to show that $N_m(T_0)$ is R-porous. Suppose that $x \in N_m(T_0)$ and V is a T_0-compatible $(i, m+2)$-good run of the game $H(A)$ such that x is its outcome and $m+1$ is a witness of Sisyfos' victory. We know that there exist $s \in \mathbb{N}$ and sequences $(n_k)_{k=1}^{\infty}$ of integers from $\{m+1, m+2, \ldots\}$, $(q_k)_{k=1}^{\infty}$ of real numbers from $(0,1)$, and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $x \in K \setminus \bigcup\limits_{n=m+1}^{\infty} S_n^{m+1}(V)$,
- $\lim\limits_{k \to \infty} n_k = \infty$,

- $\lim_{k\to\infty} q_k = 1$,
- $r_k \le 2^{-(n_k+3)} R_{n_k}$, $k \in \mathbb{N}$,
- $R^{s,q_k}_{r_k}\left(x, K \setminus S^{m+1}_{n_k}(V)\right)$, $k \in \mathbb{N}$.

We may assume that $n_k \ge \max\{i, m+2\}$ for every $k \in \mathbb{N}$. We know that the center of $B_{n+1}(V)$ lies in $d^{m+2}_n \star B_n(V)$ for every $n \ge i$ by conditions (H1) and (H2). Let us fix $k \in \mathbb{N}$. By condition (R4), we have

$$R^{s,q_k}_{r_k}\left(x, K \setminus \left(S^{m+1}_{n_k}(V) \cap B(x, 2r_k)\right)\right). \tag{3}$$

By condition (II), we have

$$N_m(T_0) \subset K \setminus \left(S^{m+1}_{n_k}(V) \cap \left(d^{m+1}_{n_k} \star B_{n_k}(V)\right)\right). \tag{4}$$

Now, let x_{n_k} be the center of $B_{n_k}(V)$, x_{n_k+1} be the center of $B_{n_k+1}(V)$, and let us take $z \in B(x, 2r_k)$. Then we have

$$\begin{aligned} d(z, x_{n_k}) &\le d(z,x) + d(x, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) < 2r_k + R_{n_k+1} + d^{m+2}_{n_k} R_{n_k} \\ &\le 2^{-(n_k+2)} R_{n_k} + 2^{-(n_k+2)} R_{n_k} + d^{m+2}_{n_k} R_{n_k} = \left(2^{-(n_k+1)} + 1 - 2^{-n_k+m+1}\right) R_{n_k} \\ &\le \left(1 - 2^{-n_k+m}\right) R_{n_k} = d^{m+1}_{n_k} R_{n_k}. \end{aligned}$$

Therefore we have $B(x, 2r_k) \subset d^{m+1}_{n_k} \star B_{n_k}(V)$, and so

$$K \setminus \left(S^{m+1}_{n_k}(V) \cap \left(d^{m+1}_{n_k} \star B_{n_k}(V)\right)\right) \subset K \setminus \left(S^{m+1}_{n_k}(V) \cap B(x, 2r_k)\right). \tag{5}$$

Finally, we have $R^{s,q_k}_{r_k}(x, N_m(T_0))$ by (3), (4), (5), and (R3). Therefore also $R^s(x, N_m(T_0))$ by (R1), (R3), and (M), and we have $R(x, N_m(T_0))$. ☐

Theorem 3.6. *Sisyfos (i.e., the second player) has a winning strategy in the game $H(A)$ if and only if the set A is σ-R-porous.*

Proof. Suppose first that $A = \bigcup_{n=1}^\infty A_n$ such that A_n is R-porous for every $n \in \mathbb{N}$. We define a strategy for Sisyfos as follows. For $n \in \mathbb{N}$ and $m \in \{1, 2, \dots, n\}$, Sisyfos plays S^m_n as the union of all balls $B \in M_n$ for which $B \subset B_n \setminus A_m$, where B_n is the nth move of Boulder.

We show that this strategy is winning. Let Boulder and Sisyfos play a run of the game $H(A)$ such that Sisyfos follows this strategy. Let x be an outcome of this run. If $x \notin A$ then Sisyfos satisfies condition (a) and wins. If $x \in A$ then there exists $m \in \mathbb{N}$ such that $x \in A_m$. Then we have $x \notin \bigcup_{n=m}^\infty S^m_n$. Further, since $R(x, A_m)$, there exists $s \in \mathbb{N}$ such that $R^s(x, A_m)$, and so we know by condition (R1) that there exist sequences $(q_k)_{k=1}^\infty$ of real numbers from $(0,1)$ and $(r_k)_{k=1}^\infty$ of real numbers from $(0, \infty)$ such that

- $\lim_{k\to\infty} q_k = 1$,
- $\lim_{k\to\infty} r_k = 0$,
- $R^{s,q_k}_{r_k}(x, A_m)$, $k \in \mathbb{N}$.

There also exists $n_0 \ge m$ such that

$$s \frac{2^{n+6} a_n}{R_{n+1}} \le \inf\{q_k \colon k \in \mathbb{N}\} \tag{6}$$

for $n \geq n_0$ since the expression on the right side is strictly positive and the expression on the left side tends to zero which follows from (2) and the estimate (derived from (1))

$$0 < s\frac{2^{n+6}a_n}{R_{n+1}} \leq s\frac{8a_n}{R_{n+2}}. \tag{7}$$

We may assume that $r_k \leq 2^{-(n_0+3)}R_{n_0}$ for every $k \in \mathbb{N}$. Let us choose $k \in \mathbb{N}$ and define n_k as the greatest integer such that

$$r_k \leq 2^{-(n_k+3)}R_{n_k}. \tag{8}$$

Obviously, we have $n_k \geq n_0$ and $\lim_{k\to\infty} n_k = \infty$. Since (8) does not hold for n_k+1 instead of n_k, we get

$$r_k > 2^{-(n_k+4)}R_{n_k+1} \geq s\frac{4a_{n_k}}{q_k} \tag{9}$$

using the estimate (6) for $n = n_k$ in the second inequality. It follows that $\frac{q_k}{2s} > \frac{2a_{n_k}}{r_k} > 0$. By condition (R2) applied to $w = \frac{2a_{n_k}}{r_k}$, we have

$$R_{r_k}^{s,q_k - s\frac{4a_{n_k}}{r_k}}\left(x, B\left(A_m, 2a_{n_k}\right)\right). \tag{10}$$

Let us denote $\tilde{q}_k = q_k - s\frac{4a_{n_k}}{r_k}$. Using the first inequality from the estimate (9), we get

$$0 \leq s\frac{4a_{n_k}}{r_k} \leq s\frac{2^{n_k+6}a_{n_k}}{R_{n_k+1}}. \tag{11}$$

By (2), (7), and (11), we have

$$\lim_{n\to\infty} s\frac{4a_{n_k}}{r_k} = 0$$

and so

$$\lim_{k\to\infty} \tilde{q}_k = \lim_{k\to\infty} q_k - \lim_{k\to\infty} s\frac{4a_{n_k}}{r_k} = 1.$$

To verify condition (b), it suffices to show that $R_{r_k}^{s,\tilde{q}_k}\left(x, K \setminus S_{n_k}^m\right)$, $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and suppose that $z \in B\left(x, 2r_k\right) \setminus B\left(A_m, 2a_{n_k}\right)$. Then

$$B\left(z, 2a_{n_k}\right) \subset K \setminus A_m \tag{12}$$

by the definition of $B\left(A_m, 2a_{n_k}\right)$. Denote the center of B_{n_k} by x_{n_k}. If we use

- Lemma 3.4 and the fact that $x \in B_{n_k+1}$ (in the second inequality of the upcoming estimate),
- an immediate consequence of (9) saying that $a_{n_k} \leq r_k$ (in the third inequality),
- estimate (8) (in the fourth inequality),

then we have for arbitrary $y \in B\left(z, 2a_{n_k}\right)$ the following:

$$\begin{aligned} d(y, x_{n_k}) &\leq d(y,z) + d(z,x) + d(x, x_{n_k}) < 2a_{n_k} + 2r_k + \left(1 - 2^{-(n_k+1)}\right)R_{n_k} \\ &\leq 4r_k + \left(1 - 2^{-(n_k+1)}\right)R_{n_k} \leq 2^{-(n_k+1)}R_{n_k} + \left(1 - 2^{-(n_k+1)}\right)R_{n_k} = R_{n_k}. \end{aligned} \tag{13}$$

This gives us the inclusion

$$B\left(z, 2a_{n_k}\right) \subset B_{n_k}. \tag{14}$$

By putting (12) and (14) together, we get $B(z, 2a_{n_k}) \subset B_{n_k} \setminus A_m$ and it easily follows from the definitions of D_{n_k} and M_{n_k} that $z \in S^m_{n_k}$. So we have $B(x, 2r_k) \setminus B(A_m, 2a_{n_k}) \subset S^m_{n_k}$ and thus

$$B(x, 2r_k) \setminus S^m_{n_k} \subset B(A_m, 2a_{n_k}). \tag{15}$$

By (10), (15), and (R3), we get $R^{s,\tilde{q}_k}_{r_k}(x, B(x, 2r_k) \setminus S^m_{n_k})$. By (R4), this gives $R^{s,\tilde{q}_k}_{r_k}(x, K \setminus S^m_{n_k})$ as we wanted.

Now, let us assume that Sisyfos has a winning strategy σ in the game $H(A)$ and that he follows this strategy in every run of the game $H(A)$. We have $A = M_0(\emptyset)$ and, by Lemma 3.5, it follows

$$A = M_0(\emptyset) \subset N_0(\emptyset) \cup \bigcup \{M_1(T_1) \colon T_1 \in \mathcal{T}\}, \tag{16}$$

where $N_0(\emptyset)$ is R-porous and $\mathcal{T}$ is an at most countable collection of $(0, 1)$-good sequences of open balls. Now, for every $T_1 \in \mathcal{T}$ we have

$$M_1(T_1) \subset N_1(T_1) \cup \bigcup \{M_2(T_1{}^\wedge T_2) \colon T_2 \in \mathcal{T}(T_1)\}, \tag{17}$$

where $N_1(T_1)$ is R-porous and $\mathcal{T}(T_1)$ is an at most countable collection of finite sequences of open balls such that $T_1{}^\wedge T_2$ is $(\mathrm{length}(T_1), 2)$-good for every $T_2 \in \mathcal{T}(T_1)$. By iterating this process, we get a countable system of R-porous sets

$$\mathcal{U} = \{N_k(T_1, T_2, \ldots, T_k) \colon k \in \mathbb{N} \cup \{0\}, T_1 \in \mathcal{T}, T_2 \in \mathcal{T}(T_1), \ldots, T_k \in \mathcal{T}(T_1, \ldots, T_{k-1})\}$$

such that for every $k \in \mathbb{N} \cup \{0\}$ and $T_1 \in \mathcal{T}, T_2 \in \mathcal{T}(T_1), \ldots, T_k \in \mathcal{T}(T_1, T_2, \ldots, T_{k-1})$, the sequence $T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_k$ is $(\mathrm{length}(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k-1}), k)$-good. It suffices to show that $A \subset \bigcup \mathcal{U}$. Suppose that this is not true and so there exists $x \in A \setminus \bigcup \mathcal{U}$. By (16), there exists $T_1 \in \mathcal{T}$ such that $x \in M_1(T_1)$. By (17), there exists $T_2 \in \mathcal{T}(T_1)$ such that $x \in M_2(T_1{}^\wedge T_2)$. In this way, we get that there exists a sequence $(T_k)_{k=1}^\infty$ where $T_1 \in \mathcal{T}$ and $T_k \in \mathcal{T}(T_1, T_2, \ldots, T_{k-1})$ for $k > 1$ such that $x \in M_k(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_k)$ for every $k \in \mathbb{N}$.

We use the sequence $(T_k)_{k=1}^\infty$ to construct a special run of the game $H(A)$. Set $S = T_1{}^\wedge T_2{}^\wedge \ldots$ The sequence S is either finite or infinite. In the first case there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $S = T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_{k_0}$ and $T_k = \emptyset$ for every $k > k_0$. Then Boulder plays balls from S and then he continues by playing open balls centered at x. Sisyfos follows his winning strategy σ. The outcome of such a run is x. Moreover, since $x \in M_{k_0}(S)$, we have $x \in \frac{1}{4} \star B_{\mathrm{length}(S)}$. It follows that the run is $(\mathrm{length}(S), m+1)$-good for every $m \in \mathbb{N}$. If the sequence S is infinite, then Boulder plays open balls following the sequence S and Sisyfos follows his winning strategy σ.

In both cases the point x is the outcome of the run and any $m \in \mathbb{N}$ cannot be a witness of Sisyfos' victory since $x \in M_m(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_m)$ and the run is $(\mathrm{length}(T_1{}^\wedge T_2{}^\wedge \ldots {}^\wedge T_m), m + 1)$-good for every $m \in \mathbb{N}$. This is a contradiction since the strategy σ is winning for Sisyfos. ☐

Lemma 3.7. *If the set A is Borel then the game $H(A)$ is Borel.*

Proof. Denote by $\mathcal{B}$ and $\mathcal{G}$ the family of all open balls in K and the family of all open subsets of K respectively. Denote the tree of all legal positions of the game $H(A)$ by $\mathfrak{T}$. Then the payoff set P for the game $H(A)$ is the set of all $V \in [\mathfrak{T}]$ ($[\mathfrak{T}]$ stands for the set of all infinite branches of $\mathfrak{T}$) of the form $V = (B_1, (S_1^1), B_2, (S_2^1, S_2^2), \dots)$ such that neither of the conditions (a) and (b) is satisfied for V. Then $[\mathfrak{T}]$ is a subset of $\prod_{n=1}^{\infty}(\mathcal{B} \times (\mathcal{G})^n)$, which will be considered as a topological space with the product topology, where each factor is equipped with the discrete topology as usual.

We define mappings $f\colon [\mathfrak{T}] \to K$ and $h_n^j\colon [\mathfrak{T}] \to \mathcal{G}$, $n \in \mathbb{N}$, $j \in \{1, 2, \dots, n\}$, by

- $\{f(V)\} = \bigcap_{n=1}^{\infty} B_n(V)$, i.e., $f(V)$ is the outcome of V,
- $h_n^j(V) = S_n^j(V)$.

It is easy to check that the mappings f and h_n^j are continuous. Next, we define

$$W_m = \{V \in [\mathfrak{T}] \colon m \text{ is a witness of Sisyfos' victory in the run } V\}. \tag{18}$$

Then we have

$$P = f^{-1}(A) \setminus \bigcup_{m=1}^{\infty} W_m.$$

The set $f^{-1}(A)$ is a continuous preimage of a Borel set and so it is Borel. To finish the proof, it remains to show that W_m is a Borel set for every $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. After taking into consideration (R1), (R3), (R5), and (M), we have $V \in W_m$ if and only if

(i) $f(V) \in K \setminus \bigcup_{n=m}^{\infty} h_n^m(V)$ and
(ii) there exists $s \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n_k \geq \max\{m, k\}$, $q_k \in (1 - \frac{1}{k}, 1) \cap \mathbb{Q}$, and $r_k \in (0, 2^{-(n_k+3)} R_{n_k}] \cap \mathbb{Q}$ such that $R_{r_k}^{s,q_k}\big(f(V), K \setminus h_{n_k}^m(V)\big)$.

Further, we have $V \in [\mathfrak{T}]$ satisfies (i) if and only if

$$V \in \bigcap_{n=m}^{\infty} \bigcup_{\substack{G \text{ is a union} \\ \text{of some balls} \\ \text{from } M_n}} \left((h_n^m)^{-1}(\{G\}) \cap f^{-1}(K \setminus G)\right).$$

The set M_n is finite, so it is easy to see that the set on the right side is closed in $[\mathfrak{T}]$. Finally, we have $R_{r_k}^{s,q_k}(f(V), K \setminus h_{n_k}^m(V))$ if and only if

$$V \in \bigcup_{\substack{G \text{ is a union} \\ \text{of some balls} \\ \text{from } M_{n_k}}} \left(\left(h_{n_k}^m\right)^{-1}(\{G\}) \cap f^{-1}\left(\{y \in K \colon R_{r_k}^{s,q_k}(y, K \setminus G)\}\right)\right)$$

and the last set is open by (R5). Thus a straightforward verification gives that W_m is Borel and we are done. $\square$

We will need the following result of J. Zapletal. To state it we need another notion of abstract porosity.

Definition 3.8 ([5])**.** Let X be a Polish space and $\mathcal{U}$ be a countable collection of its Borel subsets. An *abstract porosity* is a mapping por from all subsets of $\mathcal{U}$ to Borel subsets of X such that $\mathcal{A} \subset \mathcal{B}$ implies $\operatorname{por}(\mathcal{A}) \subset \operatorname{por}(\mathcal{B})$. The *porosity σ-ideal* associated with the porosity por is σ-generated by sets $\operatorname{por}(\mathcal{A}) \setminus \bigcup \mathcal{A}$, as $\mathcal{A}$ runs through all subsets of $\mathcal{U}$.

Theorem 3.9. [5, Theorem 4.16] *Let X be a Polish space and $\mathcal{I}$ be a porosity σ-ideal of subsets of X and $A \subset X$ be analytic. If $A \notin \mathcal{I}$, then there exists a Borel set $B \subset A$ with $B \notin \mathcal{I}$.*

Lemma 3.10. *The σ-ideal $\mathcal{I}$ of all σ-R-porous subsets of K forms a porosity ideal.*

Proof. Let $\mathcal{U}$ be a countable open basis of the space K. We define the mapping por by

$$\mathrm{por}(\mathcal{A}) = \bigl\{x \in K \colon R(x, K \setminus \bigcup \mathcal{A})\bigr\}.$$

Using the definition of R and property (R5) we get that $\mathrm{por}(\mathcal{A})$ is Borel for every $\mathcal{A} \subset \mathcal{U}$. The monotonicity of por is obvious. The verification that $\mathcal{I}$ is σ-generated by sets of the form $\mathrm{por}(\mathcal{A}) \setminus \bigcup \mathcal{A}$, $\mathcal{A} \subset \mathcal{U}$, is straightforward. □

Theorem 3.11. *Let (K, d) be a nonempty compact metric space, $R \in \mathfrak{R}(X)$, and let $A \subset K$ be an analytic set which is not σ-R-porous. Then there exists a compact set $F \subset A$ which is not σ-R-porous.*

Proof. Using Lemma 3.10 and Theorem 3.9 we may assume that A is Borel. Sisyfos does not have a winning strategy in the game $H(A)$ by Theorem 3.6. But by Theorem 3.7 and Martin Determinacy Theorem ([8]), the game is determined and so Boulder has a winning strategy μ. We consider μ as a subset of $\mathfrak{T}$ (cf. [6, 20.A]). The fact that Sisyfos has only finitely many possible choices on each of his moves of the game $H(A)$ easily implies that the body $[\mu]$ is compact in the topology derived from the topological space $\prod_{n=1}^{\infty}(\mathcal{B} \times (\mathcal{G})^n)$. Each run $V \in [\mu]$ is a run of the game $H(A)$ won by Boulder. Let $f \colon [\mathfrak{T}] \to K$ be the mapping from the proof of Theorem 3.7, that is the mapping assigning to $V \in [\mathfrak{T}]$ its outcome. Recall that the mapping f is continuous. Define $F = f([\mu])$. Then F is compact and a subset of A by condition (a) because the strategy μ is winning for Boulder.

It remains to show that F is not σ-R-porous. Since satisfaction of condition (b) does not depend on the set which the game is played with, it is obvious that μ is a winning strategy for Boulder also in the game $H(F)$. Therefore Sisyfos does not have a winning strategy in the game $H(F)$ and using Theorem 3.6 again, we get the conclusion. □

4. Applications to concrete porosities.

Using Theorem 3.11 we prove inscribing theorems for σ-porosity, σ-strong porosity, σ-strong right porosity, and σ-1-symmetrical porosity. It will be clear that Theorem 3.11 can be applied to many other types of porosity. First of all we recall definitions of the mentioned porosities.

Let (X, d) be a metric space. Let $M \subset X$, $x \in X$, and $R > 0$. Then we define

$$\begin{aligned}\theta(x, R, M) = \sup\{r > 0 \colon &\text{there exists an open ball } B(z, r) \\ &\text{such that } d(x, z) < R \text{ and } B(z, r) \cap M = \emptyset\},\\ p(x, M) = \limsup_{R \to 0+} &\frac{\theta(x, R, M)}{R}.\end{aligned}$$

We say that $M \subset X$ is

- *porous at* $x \in X$ if $p(x, M) > 0$,
- *strongly porous at* $x \in X$ if $p(x, M) \geq 1$.

Let $M \subset \mathbb{R}$, $x \in \mathbb{R}$, and $R > 0$. Then we define

$$\begin{aligned}
\theta^+(x, R, M) = &\sup\{r > 0;\ \text{there exists an open ball } B(z, r), z > x,\\
&\qquad \text{such that } |x - z| < R, \text{ and } B(z, r) \cap M = \emptyset\},\\
p^+(x, M) = &\limsup_{R \to 0+} \frac{\theta^+(x, R, M)}{R},\\
\theta^s(x, R, M) = &\sup\{r > 0;\ \text{there exists an open ball } B(z, r),\\
&\qquad \text{such that } |x - z| < R, \text{ and } (B(z, r) \cup B(2x - z, r)) \cap M = \emptyset\},\\
p^s(x, M) = &\limsup_{R \to 0+} \frac{\theta^s(x, R, M)}{R}.
\end{aligned}$$

Let $c > 0$. We say that $M \subset X$ is

- *right porous at* $x \in \mathbb{R}$ if $p^+(x, M) > 0$,
- *strongly right porous at* $x \in \mathbb{R}$ if $p^+(x, M) \geq 1$,
- *c-symmetrically porous at* $x \in \mathbb{R}$ if $p^s(x, M) \geq c$.

Theorem 4.1 (cf. [13, Theorem 3.1])**.** *Let (X, d) be a locally compact metric space. Let $A \subset X$ be a non-σ-porous analytic set. Then there exists a non-σ-porous compact set $F \subset A$.*

Proof. First, suppose that the space (X, d) is compact. Let $s \in \mathbb{N}, q \in (0, 1)$, and $r > 0$. We define a point-set relation $R_r^{s,q}$ on X by

$$\begin{aligned}
R_r^{s,q}(x, M) \Leftrightarrow\ &\text{there exists a ball } B(y, \tilde{r}) \text{ such that } x \in \big(B(y, r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{s\tilde{r}}{q})\\
&\text{and } B(y, \tilde{r}) \cap M = \emptyset.
\end{aligned}$$

We set

$$R^s = \bigcap_{q \in (0,1)} \bigcap_{R > 0} \bigcup_{0 < r < R} R_r^{s,q} \quad \text{and} \quad R = \bigcup_{s \in \mathbb{N}} R^s.$$

To show that $R \in \mathfrak{R}(X)$, we need to verify that the relations $R_r^{s,q}$, $s \in \mathbb{N}$, $r > 0$, $q \in (0, 1)$, satisfy conditions (R1)–(R5). Let us verify only (R2) and (R4), the other conditions are easy to check.

(R2) Let $s \in \mathbb{N}$, $r > 0$, $q \in (0, 1)$, $M \subset X$, $x \in X$, $0 < w < \frac{q}{2s}$, and suppose that $R_r^{s,q}(x, M)$. There exists an open ball $B(y, \tilde{r})$ such that

$$x \in \big(B(y, r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{s\tilde{r}}{q}) \quad \text{and} \quad B(y, \tilde{r}) \cap M = \emptyset.$$

So we have

$$\frac{s\tilde{r}}{q} > d(x, y) > \frac{r}{2} \tag{19}$$

and so $\tilde{r} - rw > r\left(\frac{q}{2s} - w\right) > 0$. Clearly, $B(y, \tilde{r} - rw) \cap B(M, rw) = \emptyset$ and by (19) we have

$$s\frac{\tilde{r} - rw}{q - 2sw} > s\frac{\tilde{r}(1 - \frac{2sw}{q})}{q - 2sw} = \frac{s\tilde{r}}{q} > d(x, y).$$

Thus $x \in B\big(y, s\frac{\tilde{r}-rw}{q-2sw}\big)$ and we can conclude that $R_r^{s,q-2sw}(x, B(M, rw))$.

(R4) Let $s \in \mathbb{N}$, $r > 0$, $q \in (0, 1)$, $M \subset X$, and $x \in X$ be such that $R_r^{s,q}(x, M \cap B(x, 2r))$. Then there exists an open ball $B(y, \tilde{r})$ such that

$$x \in \big(B(y, r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{s\tilde{r}}{q}) \quad \text{and} \quad B(y, \tilde{r}) \cap M \cap B(x, 2r) = \emptyset.$$

First, let us assume that $\tilde{r} \le r$. If $z \in B(y, \tilde{r})$ then

$$d(z, x) \le d(z, y) + d(y, x) < \tilde{r} + r \le 2r.$$

So we have $B(y, \tilde{r}) \subset B(x, 2r)$ and therefore $B(y, \tilde{r}) \cap M = B(y, \tilde{r}) \cap M \cap B(x, 2r) = \emptyset$. It follows that $R_r^{s,q}(x, M)$. Now, let us assume that $\tilde{r} > r$. Then we have

$$B\,(y, r) \cap M = B\,(y, r) \cap M \cap B\,(x, 2r) \subset B(y, \tilde{r}) \cap M \cap B\,(x, 2r) = \emptyset$$

and the open ball $B\,(y, r)$ witnesses that $R_r^{s,q}\,(x, M)$. The opposite implication in (R4) is obvious.

It is also straightforward to verify that $M \subset X$ is porous at $x \in X$ if and only if M is R-porous at x. Therefore, A is not σ-R-porous and by Theorem 3.11, there exists a non-σ-R-porous (and thus also non-σ-porous) compact set $F \subset A$.

Now, suppose that (X, d) is an arbitrary locally compact metric space. Since A is a non-σ-porous subset of X, there exists $x \in X$ such that $A \cap B(x, r)$ is a non-σ-porous subset of X for every $r > 0$ by Theorem 2.2. Let us take $r_0 > 0$ such that $\overline{B\,(x, r_0)}$ is compact and denote $A' = A \cap B(x, r_0)$. Since porosity is a local property, every $M \subset B(x, r_0)$ is σ-porous in X if and only if M is σ-porous in the compact metric space $\overline{B\,(x, r_0)}$. Therefore, A' is non-σ-porous in $\overline{B\,(x, r_0)}$. Due to the previous part of the proof, there exists a non-σ-porous (in $\overline{B\,(x, r_0)}$ and therefore also in X) compact set $F \subset A' \subset A$. □

Theorem 4.2. *Let (X, d) be a locally compact metric space. Let $A \subset X$ be a non-σ-strongly porous analytic set. Then there exists a non-σ-strongly porous compact set $F \subset A$.*

Proof. Similarly as in the previous proof we may assume that X is compact. Let $q \in (0, 1)$ and $r > 0$. We define a point-set relation R_r^q on X by

$$\begin{aligned} R_r^q(x, M) \Leftrightarrow{} & \text{there exists a ball } B(y, \tilde{r}) \text{ such that } x \in \big(B(y, r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{\tilde{r}}{q}) \\ & \text{and } B(y, \tilde{r}) \cap M = \emptyset. \end{aligned}$$

We set

$$R = \bigcap_{q \in (0,1)} \bigcap_{R > 0} \bigcup_{0 < r < R} R_r^q.$$

One can easily check that $R \in \mathfrak{R}(X)$. Then $M \subset X$ is σ-strongly porous if and only if A is σ-R-porous. Applying Theorem 3.11, we get the conclusion. □

Theorem 4.3. *Let $A \subset \mathbb{R}$ be a non-σ-strongly right porous analytic set. Then there exists a non-σ-strongly right porous compact set $F \subset A$.*

Proof. Without any loss of generality, we may assume that $A \subset (0,1)$. Let $q \in (0,1)$ and $r > 0$. We define a point-set relation R_r^q on $[0,1]$ by

$$\begin{aligned} R_r^q(x,M) \Leftrightarrow \text{ there exist } y \in \mathbb{R}, \tilde{r} > 0 \text{ such that } y > x, x \in \big(B(y,r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{\tilde{r}}{q}) \\ \text{and } B(y,\tilde{r}) \cap M = \emptyset. \end{aligned}$$

We set

$$R = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0<r<R} R_r^q.$$

One can easily check that $R \in \mathfrak{R}([0,1])$. Then $M \subset (0,1)$ is σ-strongly right porous if and only if M is σ-R-porous. Applying Theorem 3.11 we get the conclusion. ☐

Remark 4.4. Theorem 4.3 has been already used in [7].

Theorem 4.5. *Let $A \subset \mathbb{R}$ be a non-σ-1-symmetrically porous analytic set. Then there exists a non-σ-1-symmetrically porous compact set $F \subset A$.*

Proof. Without any loss of generality, we may assume that $A \subset (0,1)$. Let $q \in (0,1)$ and $r > 0$. We define a point-set relation R_r^q on $[0,1]$ by

$$\begin{aligned} R_r^q(x,M) \Leftrightarrow \text{ there exist } y \in \mathbb{R}, \tilde{r} > 0 \text{ such that } x \in \big(B(y,r) \setminus \overline{B}(y, \tfrac{1}{2}r)\big) \cap B(y, \tfrac{\tilde{r}}{q}) \\ \text{and } \big(B(y,\tilde{r}) \cup B(2x-y,\tilde{r})\big) \cap M = \emptyset. \end{aligned}$$

We set

$$R = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0<r<R} R_r^q.$$

We can easily verify that $R \in \mathfrak{R}([0,1])$ and that $M \subset (0,1)$ is σ-1-symmetrically porous if and only if M is σ-R-porous. The rest of the proof follows from Theorem 3.11. ☐

Finally, we apply Theorem 4.5 to answer a question posed by M. J. Evans and P. D. Humke in [4]. This is the following question.

Question. *Does there exist an F_σ set in $[0,1]$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous?*

We answer this question positively by proving the next theorem.

Theorem 4.6. *There exists a closed set $F \subset [0,1]$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous.*

Proof. There exists a Borel set $A \subset (0,1)$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous ([3]). By Theorem 4.5, there exists a compact non-σ-1-symmetrically porous set $F \subset A$. Since F is a subset of A, it is still σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$. ☐

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