Synthetic approach to Chasles theorem for timelike ruled surface

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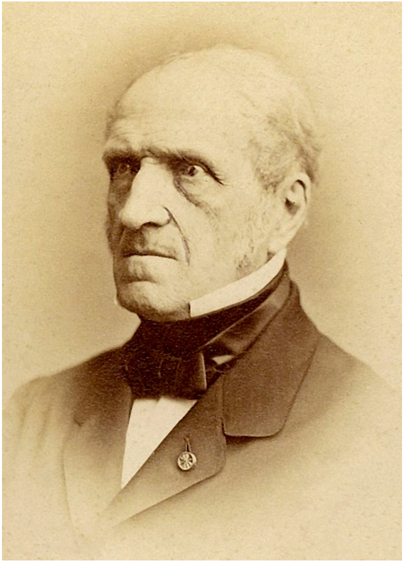
WDS–M
Prague
8.6.2015

Michel Chasles



Michel Floréal Chasles

1793-1880

French mathematician

1841 Professor at the École Polytechnique

1846 chair of higher geometry at the Sorbonne

Works

- 1837 Aperçu historique sur l'origine et le développement des méthodes en géométrie
- 1852 Traité de géométrie
- 1865 Traité des sections coniques

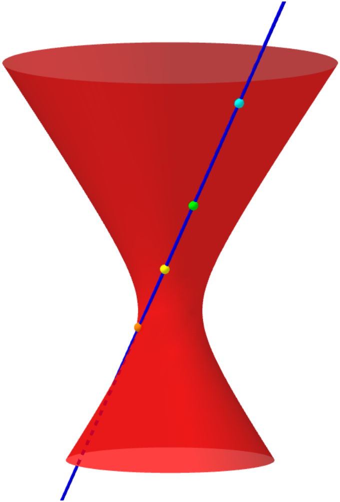
Chasles theorem (for timelike ruled surface)

Correspondence Mathématique et Physique XI (Bruxelles 1839), p.53

Donc :

Quatre plans tangens à une surface gauche, menés par une même génératrice, ont leur rapport anharmonique égal à celui de leurs quatre points de contact.

Let Σ be a non-developable timelike ruled surface, and l be a generatrix of Σ. Then points on l are in projectivity with tangent planes to Σ in these points.



Through every point of Σ there is a straight line that lies on Σ

ruling line = generatrix

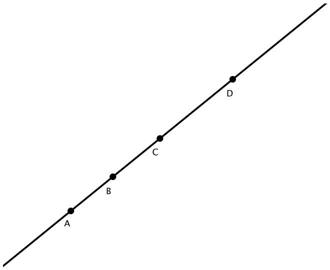
finite number of singular (torsal) lines

2nd degree surfaces:

hyperboloid

hyperbolic paraboloid

Cross-ratio

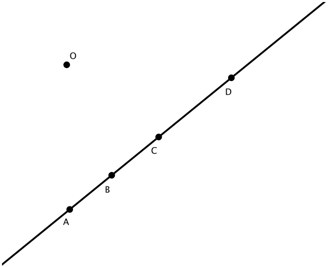


A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

oriented („length“ of) line segment

Cross-ratio

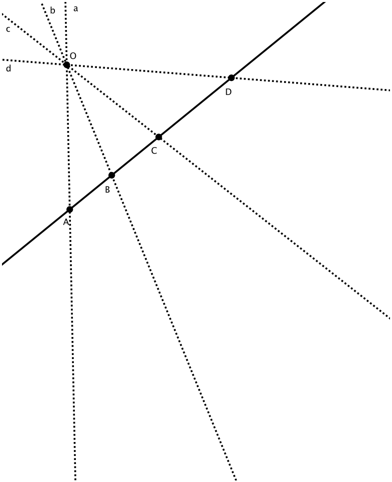


A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

Projection from point O

Cross-ratio



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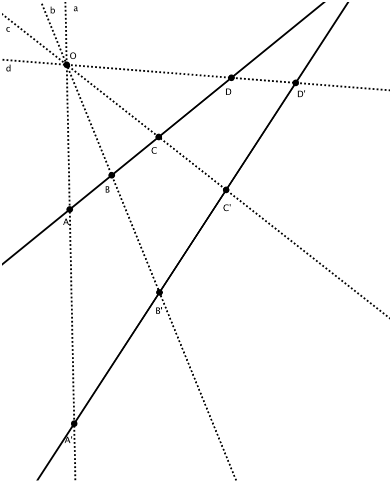


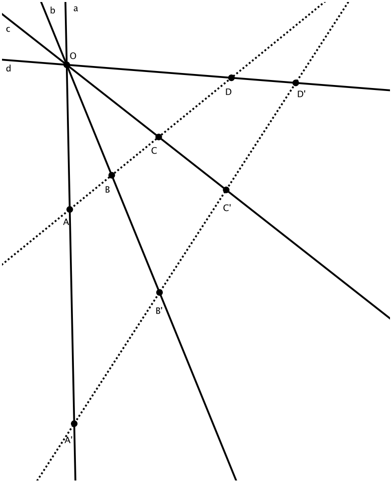
A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

Projection from point O

$$(A, B; C, D) = (A', B'; C', D')$$

Cross-ratio is projective invariant



A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

Projection from point O

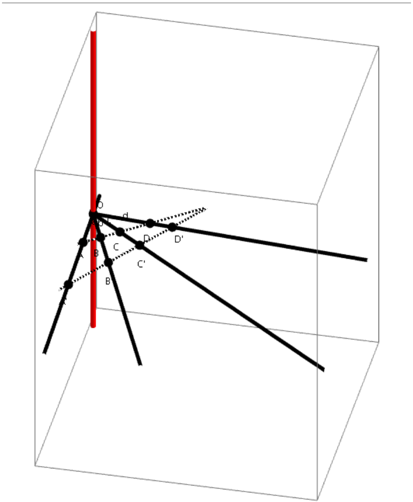
$(A, B; C, D) =$
$(A', B'; C', D')$

Cross-ratio is projective invariant

Pencil of lines a, b, c, d

$(A, B; C, D) = (a, b; c, d)$

Cross-ratio



A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

Projection from point O

$$(A, B; C, D) = (A', B'; C', D')$$

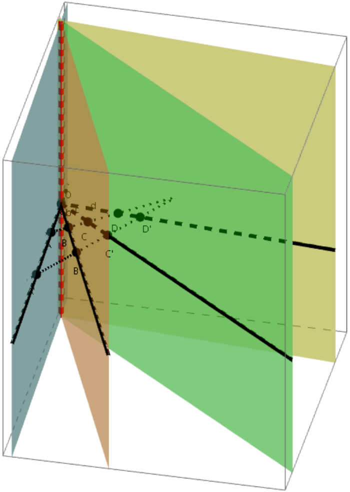
Cross-ratio is projective invariant

Pencil of lines a, b, c, d

$$(A, B; C, D) = (a, b; c, d)$$

3rd dimension

Cross-ratio



A, B, C, D collinear points

$$(A, B; C, D) = \frac{\frac{AC}{BC}}{\frac{AD}{BD}}$$

AC – oriented length

Projection from point O

$$(A, B; C, D) = (A', B'; C', D')$$

Cross-ratio is projective invariant

Pencil of lines a, b, c, d

3rd dimension

$$(A, B; C, D) = (a, b; c, d) = (\alpha, \beta; \gamma, \delta)$$

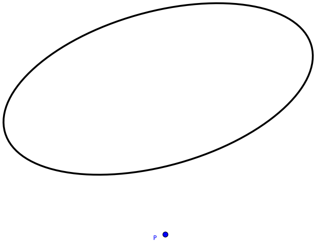
Pole and polar

Conic: $x \cdot Q \cdot x^T =$
$a_{1,1}x_1^2 + 2a_{1,2}x_1x_2 + a_{2,2}x_2^2 + 2a_{1,0}x_1x_0 + 2a_{2,0}x_2x_0 + a_{0,n}x_0^2 = 0$

$$Q = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,0} \\ a_{2,1} & a_{2,2} & a_{2,0} \\ a_{0,1} & a_{0,2} & a_{0,n} \end{pmatrix}$$

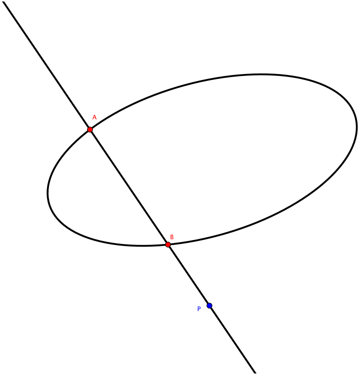
Pole: $p = \begin{pmatrix} p_1 & p_2 & p_0 \end{pmatrix}$

Polar: $p \cdot Q \cdot x^T = 0$



Pole and polar (synthetically)

Find harmonic conjugate
to P on any secant



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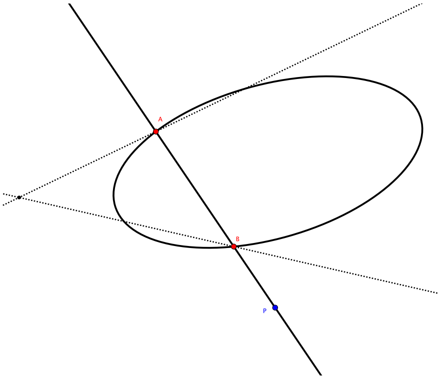

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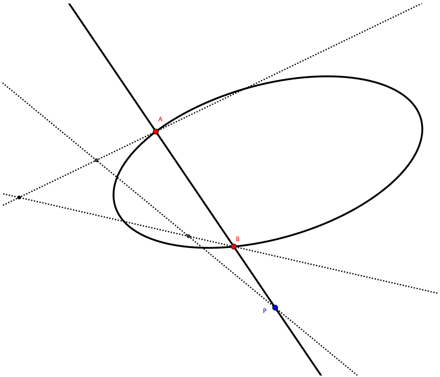

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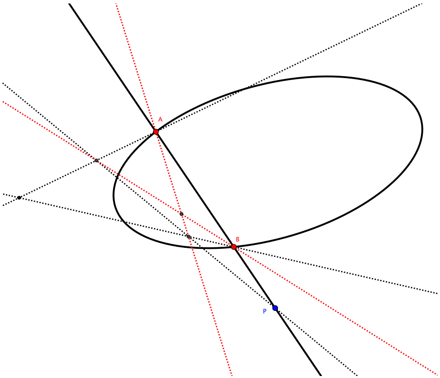
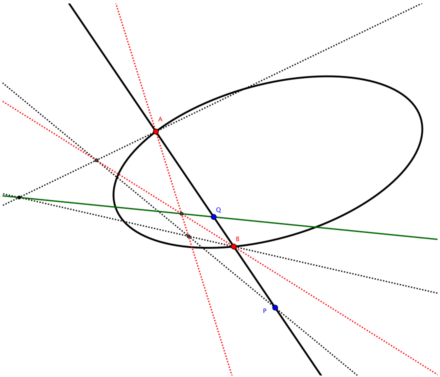


Pole and polar (synthetically)

Find harmonic conjugate
to P on any secant

$(A, B; P, Q) = -1$



Pole and polar (synthetically)

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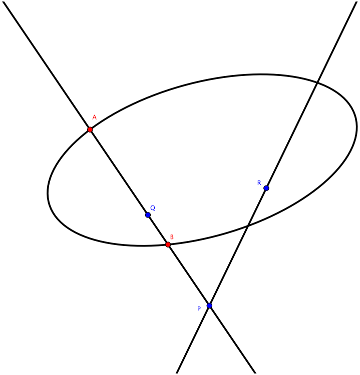
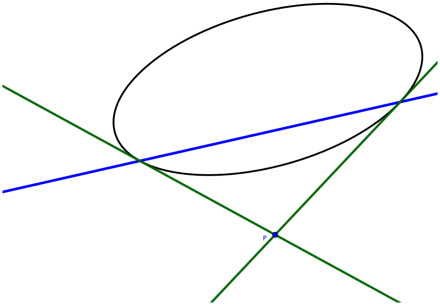


Pole and polar (synthetically)

Find harmonic conjugate to P on any secant

$(A, B; P, Q) = -1$

polar intersects conic in tangent points



Pole and polar plane

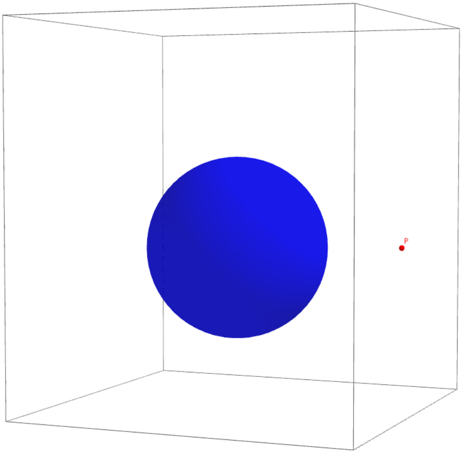
$$Q = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,0} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,0} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,0} \\ a_{0,1} & a_{0,2} & a_{0,3} & a_{0,n} \end{pmatrix}$$

Quadric: $x \cdot Q \cdot x^T = 0$

Pole:

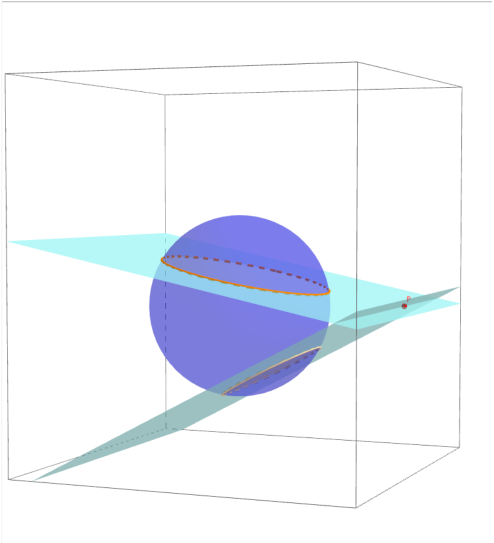
$p = \begin{pmatrix} p_1 & p_2 & p_3 & p_0 \end{pmatrix}$

Polar plane: $p \cdot Q \cdot x^T = 0$



Pole and polar plane (synthetically)

Find polars on secant planes through P



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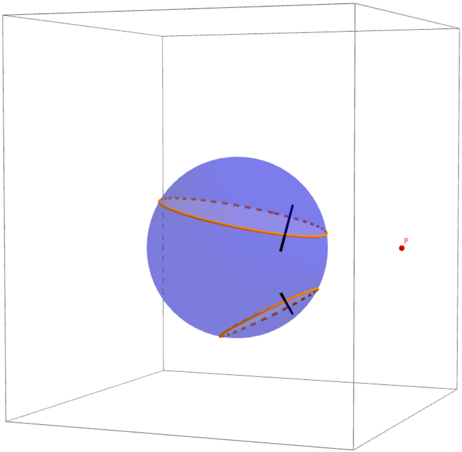
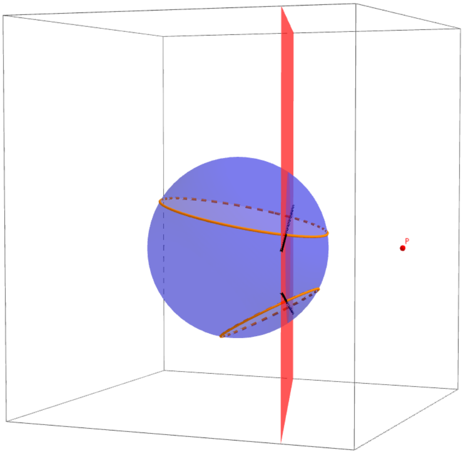


Pole and polar plane (synthetically)

Find polars on secant planes through P

Plane through polars

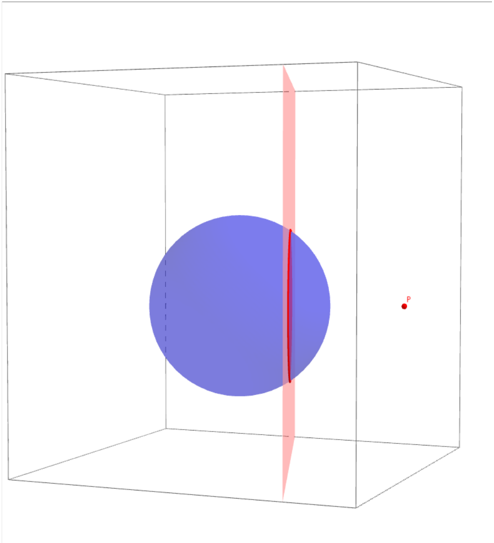


Pole and polar plane (synthetically)

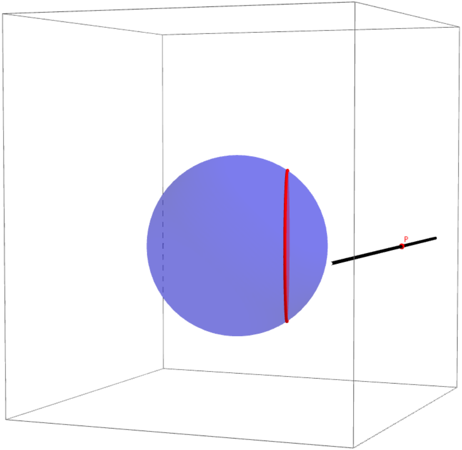
Find polars on secant planes through P

Plane through polars

Polar plane intersects quadric in tangent points



Polar planes to points on a line

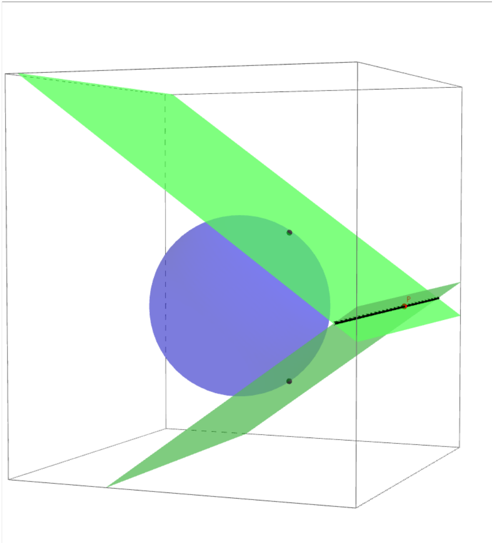


Surface degree and number of tangent planes

Polar planes to points on a line

Tangent planes through line to a quadric

Surface of 2nd degree has 2 tangent planes

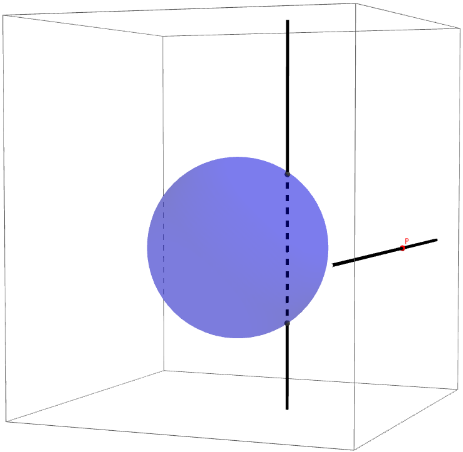


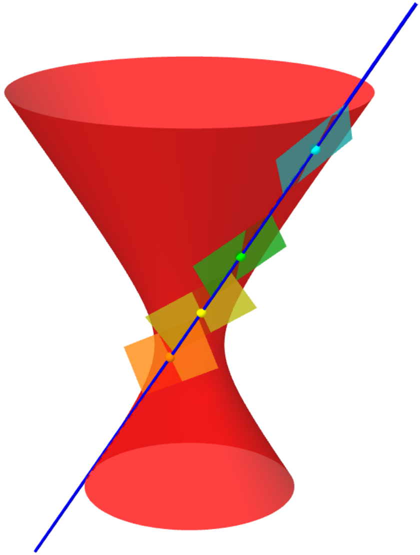
Polar planes to points on a line

Tangent planes through line to a quadric

Surface of 2nd degree has 2 tangent planes

Polar planes to points through given line rotates around conjugate polar of this line



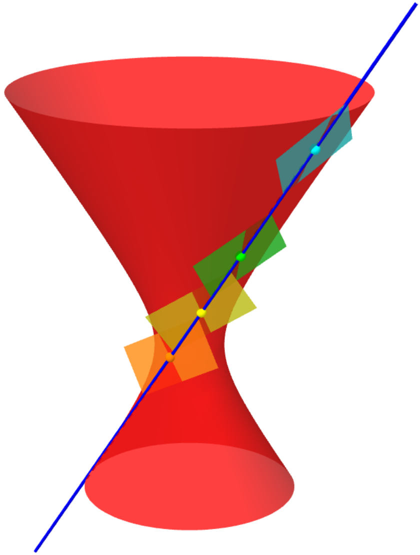


Polar plane to point on surface is tangent plane

Ruling line is self-conjugate

Tangent planes are in projectivity with their contact points

Chasles theorem for 2nd degree surfaces



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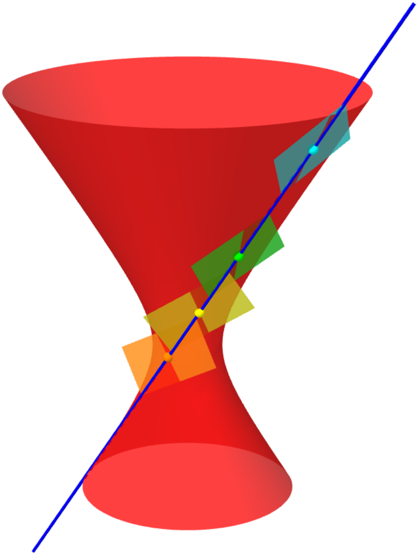
Chasles theorem for 2nd degree surfaces

Timelike ruled surface of 2nd degree |GGB



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Chasles theorem for 2nd degree surfaces

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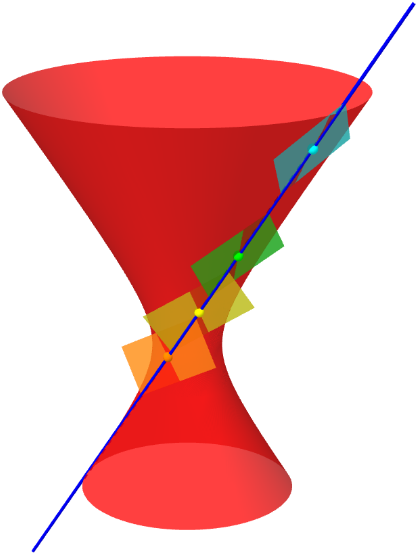


Polar plane to point on surface is tangent plane

Ruling line is self-conjugate

Tangent planes are in projectivity with their contact points

Chasles theorem for 2nd degree surfaces

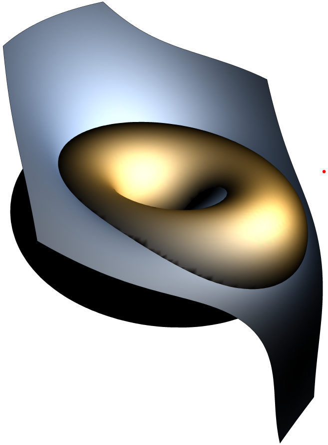
General (algebraic) surface

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_0 \end{pmatrix}$$
$$f(x) = 0$$

form of n–th degree

Pole:
$$p = \begin{pmatrix} p_1 & p_2 & p_3 & p_0 \end{pmatrix}$$

1st polar: $\sum_{i=0}^{3} \frac{\partial f(x)}{\partial x_i} p_i$



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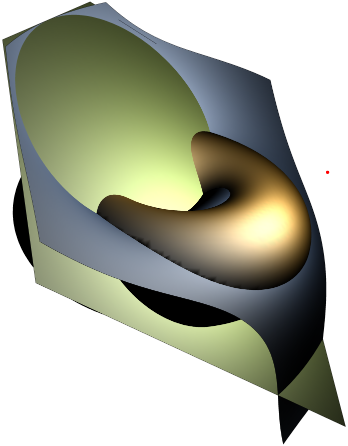
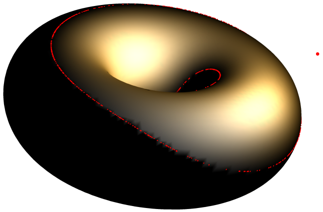
1st polar: $\sum_{i=0}^{3} \frac{\partial f(x)}{\partial x_i} p_i$

last polar: $\sum_{i=0}^{3} \frac{\partial f(p)}{\partial p_i} x_i$



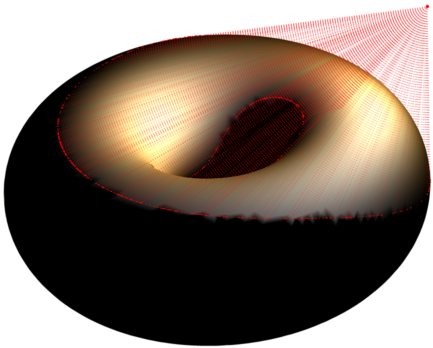
General (algebraic) surface

1st polar intersects
surface in tangent points



1st polar intersects
surface in tangent points

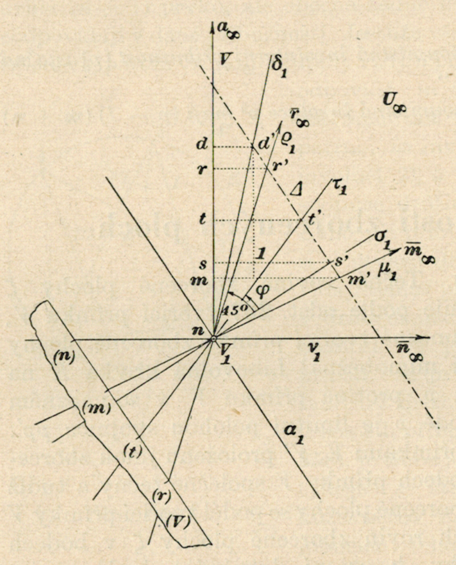
curve of intersection =
edge of the shadow



Given 3 points on the generatrix with 3 tangent planes

Simple construction of tangent plane through any other point.

Projectivity of collinear points and pencil of lines can be done with strip of paper



Obr. 659.

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Thank you for your attention