NMAI059 Probability and Statistics
Exercise assignments and supplementary examples
October 30, 2017

How to use this guide. This guide is a gradually produced text that will contain key exercises to practise discussed topics. The attend of this text is to make the practicals follow the lectures as much as possible. However, it is not always possible regarding the schedule and the holidays.

The assignments of exercises for each week are not thought to be entirely discussed during the practicals. Only a part of the prepared exercises will be covered there. The rest should serve for the individual practise, preparation for the exam, discussion with classmates, ...

1. Week 2.10–6.10.

Exercise 1.1. Gradually calculate and perform the experiments:

(1) What is the probability of rolling a six on a dice?
(2) Roll a hexahedral dice. How many of you rolled a six? Was it the expected result?
(3) Roll the dice ten times. How many sixes did you get together now?
(4) Repeat the experiment distinguishing the odd/even numbers.
(5) Repeat the experiment distinguishing the divisible/indivisible numbers.
(6) Which result is the closest to the expectation?
(7) Repeat the experiments how many times you want and think about the results.

Exercise 1.2. Again, you will roll the hexahedral dice. This time, write down the order of the try, in which you obtain a given event for the first time.

(1) In which try did you roll six? What is the maximal number obtained in the classroom?
(2) Repeat this experiment few times and write down the results. How many times did you roll six in the first try, in the second try, in the third try etc.
(3) Especially, be focused on the maximal and minimal recorded values.
(4) Are those achieved values in accordance with your expectations?
(5) Repeat the experiment, but instead of six you wait for the first even number, the first number divisible by three, the first number greater than three and so on. Watch the differences in each experiment.
(6) What are the theoretical probabilities of the considered events?

Try to find some random number generator and perform the experiments using this generator. Are the results in accordance with the results obtained by real experiments?

Exercise 1.3. Invent a suitable generator of a random sequence of zeros and ones using a hexahedral dice.

(1) Try to make up a random sequence of zeros and ones of the length 100.
(2) Generate a random sequence of zeros and ones using the generator that you invented.
(3) Use some generator of random numbers available via computer (phone, ...).
(4) Which sequence do you find “more random” and why?

Exercise 1.4. Recall the basic propositions concerning combinations, permutations and calculation of discreet probability discussed in the previous lectures.

2. Weeks 9.10–20.10.

2.1. **Classical Probability.**

Exercise 2.1. We roll four dice. What is the probability that

(1) four different numbers were thrown,
(2) only odd numbers were thrown,
(3) the sum of the four thrown numbers is equal to 6,
(4) the sum of the thrown numbers is greater that 5?

Exercise 2.2. What is the probability that at least one six is thrown if we roll

(1) two dice,
(2) n dice.

Exercise 2.3. Consider n different letters and n different envelopes (with already prescribed address). A confused secretary inserts the letters into the envelopes completely randomly.

(1) What is the probability that at least one letter is in the right envelope?
(2) What is the probability that none of the letters is in the right envelope? Compute the limit of the probability for $n \to \infty$.

In the following, the term of distinguishability and indistinguishability appears. It can lead to various speculations of what does the term mean. Imagine two balls and two boxes, each ball is randomly assigned to one box. In this case, we have three different states: $(2,0),(1,1),(0,2)$. In the distinguishable case, the probabilities of the states are $1/4, 1/2, 1/4$, because the second state actually includes two states—depending on which ball is in which box. In the indistinguishable case, all states are equal and therefore, the probabilities are always $1/3$. It is a question of the model of randomness that plays the role. Note that in common applications, we usually have distinguishable balls.

Exercise 2.4 (Maxwell-Boltzmann scheme). Consider r **distinguishable** objects and n boxes. The objects are randomly placed into the boxes while all placements are equally probable. Determine

(1) the probability that a given box contains exactly k objects,
(Hint: Consider the ordered r-tuples of numbers $1 \ldots n$ declaring in what box a given object is.)
(2) the limit of this probability for $n \to \infty$, $r \to \infty$ such that $r/n \to \lambda > 0$,
(3) the probability that no box is empty.
(Hint: It is easier to compute the probability of the complementary event.)

Exercise 2.5 (Bose-Einstein scheme). Consider r **indistinguishable** objects and n boxes. The boxes are randomly places into the boxes while all placements are equally probable. Determine

(1) the probability that a given box contains exactly k objects,
(Hint: Consider a suitable "graphical" representation.)
(2) the limit of the probability for $n \to \infty$, $r \to \infty$ such that $r/n \to \lambda > 0$,
(3) the probability that no box is empty.
(Hint: Compute directly the probability of this event and again use a suitable "graphical" representation.)

Exercise 2.6.
Sixteen passengers got on a train with ten carriages and chose the carriage randomly. Determine the probability that there is at least one passenger in each carriage.
(Hint: One of the previous schemes can be used.)

2.2. **Conditioning.**

Exercise 2.7. A game consisting of five rounds has the following rules: You win or lose the first round with the same probability (1/2). The probability of winning changes in each following round. If you won the previous round, the probability of winning increases by $1/10$ compared to the previous round. If you lost the previous round, the probability of winning decreases by $1/10$ compared to the previous round.

(1) What is the probability that you win at least three rounds?
(2) What is the probability that the probability of winning in the fifth round is $0,7$?
(3) What is the probability that you win in the second round, if the probability of winning in the third round is $0,5$?

Exercise 2.8. We roll two regular dice.

(1) What is the probability that a six was thrown given that the sum is 8?
(2) Are the events [a six was thrown] and [the sum is 8] independent?
(3) What is the probability that a six was thrown on the first dice given that a six was thrown at least on one dice.

Exercise 2.9. We roll two regular dice. — one is blue and one is green. Denote the events A=[an even number was thrown on the blue dice], B =[an odd number was thrown on the green dice], C =[the sum of the numbers is odd]. Are the events A, B, C pairwise independent? Are the events A,B,C independent?

Exercise 2.10. A tennis player has the success rate 60% in the first service and 80% in the second service.

(1) What is the probability of double fault?
(2) What is the probability of a fault in the first service given that a double fault did not follow.

Exercise 2.11. A classroom consists of 70% of boys and 30% of girls. 10% of boys wears long hair and 80% of girls wears long hair.

(1) What is the probability that a randomly chosen person has long hair?
(2) A chosen person has long hair. What is the probability that the person is a girl?

Exercise 2.12. A random number of coins lies on a table: the probability that there is exactly k coins on the table is equal to $2/3^k$ for $k = 1, 2, \ldots$. We will toss the coins at one time. If there is a tail on every coin, then we get a treat (and be happy).

(1) Is it more probable that we get the treat or not?
(2) If we did not get the treat, what is the probability that there was exactly n coins on the table?

Exercise 2.13. There are N plates with little cakes on them in the kitchen. On the i-th plate, there is a_i cakes with quark filling and b_i cakes with plum jam filling, which can not be distinguished by the naked eye. Therefore, we randomly choose one plate and one cake from this plate. We find out that the cake has plum jam filling. What is the probability that we choose k-th plate?

Important model for the classical probability is *Pólya urn scheme*. It models picking out balls with different colors from a urn. The rules are following:

(1) At the beginning, there are n balls of k different colors, specifically n_i balls of the color i.
(2) In each round, we pick out one ball from the urn. Each ball is chosen with the same probability.
(3) After picking out a ball and writing down the color of the ball, we return the ball to the urn together with d additional balls of the same color. Important cases: we denote by $d = -1$ the *picking out without returning*, by $d = 0$ the *picking out with returning*

Exercise 2.14. There are 5 white balls and 5 black balls in the urn. Determine the probability of picking out a white ball in the k-th round

(1) having $d = -1$.
(2) having $d = 0$.
(3) having $d = 1$.

Take into consideration general starting number of white balls and black balls in the urn.

Exercise 2.15. Let us have three urns each with four balls. In the first urn, there are three black balls and one white ball. In the second urn, all balls are white. In the third urn, there are again three black balls and one white ball.

(1) Randomly pick one urn. What is the probability that you choose white ball from this urn?
(2) Randomly pick one urn. A ball chosen from this urn is white. What is the probability that the urn contains only white balls?
(3) Randomly pick one urn. A ball chosen from this urn is white. We do not return the ball into the urn. What urn should we now choose so the picking out of a black ball would be the most probable?

Exercise 2.16. There is one white ball and one black ball in a urn. We perform n rounds of choosing balls from this urn. After each round, we return the ball back into the urn together with one additional ball of the same color. Show that the probability of i white balls in the urn after n rounds is the same for all $i = 1, 2, \ldots, n-1$.

Exercise 2.17. Consider two symmetrical coins. We are interested in the result of two heads being tossed at the same time. Determine

(1) What is the probability that you toss two tails at the same time before you toss two heads at the same time?
(2) What is the probability that you toss two heads in the fourth round at the latest?
(3) What is the probability that you toss two heads for the first time in the k-th round given that you tossed two heads for the second time in the sixth round?

Exercise 2.18. Cyril and Dana are playing a simple game. If the number on the thrown dice is smaller or equal to 4, Cyril pays 1 dollar to Dana. If the number is 5 or 6, then Dana pays 1 dollar to Cyril. In the beginning, Cyril has 2 dollars and Dana has 1 dollar. The game ends if one of the players has nothing. The other one wins.

(1) What is the probability that Cyril wins?
(2) What is the probability that Cyril wins if he had 4 dollars in the beginning and Dana had 2 dollars?

Exercise 2.19. A dentist placed *inexhaustibly many* chocolate candies of *five* different flavours into the bowl in the waiting room. Each of the eight patients randomly choose one flavour independently of the others (*randomly means that each flavour is chosen with the same probability*).

(1) What is the probability that each flavour was chosen by at least one patient?
(2) What is the probability that all patients agreed on the same flavour?
(3) What is the probability that each flavour was chosen by at least one patient if there are c different flavours and z patients?
(4) What is the probability that exactly k flavours were chosen, where $k = 1, \ldots, 5$ ($c = 5$ and $z = 8$)?

(*It can be suitable to use* $P(A) = 1 - P(A^C)$ *at some point.*)

2.3. **A couple of problematic exercises.**

Exercise 2.20. Let A and B be independent events. Show that also A^C and B^C are independent.

Generalize for an arbitrary number of events in the sense that if $A_1, \ldots A_n$ are independent, then $B_1, \ldots, B_n$ are independent as well, where B_i is either A_i or A_i^C.

Exercise 2.21. Let A and B are two random events such that $\mathrm{P}(A) > 0, \mathrm{P}(B) > 0$. Prove or disprove the following:

(1) A and B are independent $\Rightarrow \mathrm{P}(A|B) = \mathrm{P}(A)$.
(2) $\mathrm{P}(A|B) = \mathrm{P}(A) \Rightarrow A$ and B are independent.
(3) $A \cap B = \emptyset \Rightarrow A$ and B are independent.
(4) $A \cap B \neq \emptyset \Rightarrow A$ and B are independent.
(5) A and B are independent $\Rightarrow A \cap B = \emptyset$.
(6) A and B are independent $\Rightarrow A \cap B \neq \emptyset$.
(7) $\mathrm{P}(A|B) = \mathrm{P}(A|B) \Rightarrow A$ and B are independent.
(8) $\mathrm{P}(A|B) = \mathrm{P}(A|B) \Rightarrow A$ and B are independent.
(9) Every event A is independent of an event B, if $\mathrm{P}(B) = 1$ holds.
(10) If $\mathrm{P}(A) = 0$ holds, then A is independent of every event B.

3. Week 23.10–27.10.

3.1. **Random variables, vectors and its distribution.**

Exercise 3.1. There are b white balls, m blue balls and c red balls placed in a urn and $b+c+m=\ell$. Each ball is picked up with equal probability. We choose n balls and write down the colors. Determine the probability distribution of:

(1) the number of red balls being chosen in case of choosing without returning and $n=10$, $c=6$, $b=8$, $m=6$.
(2) the number of red balls being chosen in case of choosing with returning and $n=10$, $c=6$, $b=8$, $m=6$.
(3) the number of white and red balls being chosen (now we talk about a random vector) in case of choosing without returning and $n=10$, $c=6$, $b=8$, $m=6$.
(4) the number of white and red balls being chosen (now we talk about a random vector) in case of choosing with returning and $n=10$, $c=6$, $b=8$, $m=6$.

In the first two cases, did you obtain the marginal distributions of the second two distributions? Perform this try experimentally (using a random number generator or with drawing lots). How do you deal with the choosing without returning when you use the random number generator?

Exercise 3.2. The time of waiting for a response (measured in hours) has the probability density

$$f(x)=\begin{cases} e^{-x} & x>0 \\ 0 & x<0. \end{cases}$$

We expect $n=5$ responses while the times of waiting are independent. Determine:

(1) What is the distribution of the number of responses that arrive within one hour?
(2) What is the distribution of the number of responses that arrive within two hours?
(3) What is the probability that no response will arrive within half an hour?
(4) What is the distribution of the time of waiting for the last response?
(5) Is it possible to claim some limit statement at points (3) and (4) considering $n\to\infty$?

Exercise 3.3. Let X be a random variable with uniform distribution on the interval $(1,6)$. Determine:

(1) the area distribution of a circle with radius X.
(2) the area distribution of a square with the side of the length X.
(3) the area distribution of an equilateral triangle with the side of the length X.
(4) the area distribution of a cube with the side of the length X.

Consider also discrete uniform distribution on the set $\{1,2,\dots,6\}$. It can be modelled empirically using a hexahedral dice. Is there a better way to get closer to the distributions in the cases (1)–(4) using a dice (one or more)?

Exercise 3.4. Roll a hexahedral dice repetitively. Construct an empirical distribution function and compare it with the theoretical one. Do you find your dice complying or not? Does the consistency/inconsistency of the empirical and the theoretical distribution functions change with an increasing number of tosses?

Exercise 3.5. Consider that you toss two hexahedral dice and write down the result K_1, K_2. Determine:

(1) the distribution of the sum K_1+K_2.
(2) the distribution of the absolute difference $|K_1-K_2|$.
(3) the joint distribution $(K_1+K_2, |K_1-K_2|)$. Are these random variables independent?
(4) Perform an experiment and observe how the empirical distribution functions look like.

Exercise 3.6. Toss one hexahedral dice and write down the result S. Then toss n dice and sum the results into one random variable V while: $n=1$ if $S\in\{1,2,3\}$, $n=2$ if $S\in\{4,5\}$ and $n=3$ if $S=6$. Determine:

(1) the distribution of V, i.e. the sum of the results on all dice (excluding the result S obviously).
(2) the joint distribution of V and S. Are these random variables independent?
(3) Watch the consistency/inconsistency of the empirical distribution function of V with the theoretical one using an experiment. Can you compare the empirical and the theoretical distribution functions of the couple (V, S) in a similar way?

4. Week 30.10–3.11.

4.1. **Joint and marginal distributions.**

Exercise 4.1. Let us have two random variables X and Y with the same discrete distribution. Find possible joint distributions:

(1) if $\mathrm{P}[X = 0] = \mathrm{P}[X = 1] = 1/2$.
(2) if $\mathrm{P}[X = 0] = 1 - \mathrm{P}[X = 1] = p$
(3) if $\mathrm{P}[X = 0] = p_0$, $\mathrm{P}[X = 1] = p_1$, $\mathrm{P}[X = 2] = 1 - p_1 - p_2$.

Exercise 4.2. Consider two random variables X and Y with the same continuous distribution. Find possible joint distributions:

(1) if X and Y have uniform distribution on $[0, 1]$.
(2) if X and Y have probability density $f(x) = 2x$ for x on $[0, 1]$ and 0 otherwise.

How many such distributions do you think is possible to find?

Exercise 4.3. Consider a two-dimensional discrete random vector $\boldsymbol{X} = (X, Y)$ with a probability density defined by:

$$p(\boldsymbol{x}) = \frac{1}{6}, \boldsymbol{x} \in \{(0, 2), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}.$$

(1) Determine the marginal distributions corresponding to the joint distribution.
(2) Does the value of X affect the behaviour of Y?
(3) Will the answers to the previous two questions change if the joint distribution is given by the table

$Y \backslash X$	0	1	2
0	1/36	1/18	1/12
1	1/18	1/9	1/6
2	1/12	1/6	1/4

Exercise 4.4. Consider a two-dimensional continuous random vector $\boldsymbol{X} = (X, Y)$ with a probability density defined by

$$p(\boldsymbol{x}) = 2, \boldsymbol{x} \in \{[0, 1]^2, x + y > 1\}.$$

(1) Determine the distribution function of this distribution.
(2) Determine the marginal distribution corresponding to the joint distribution (both density and distribution function).
(3) Does the value of X affect the behaviour of Y?
(4) Will the answers to the previous two questions change if the joint distribution is defined by

$$f(\boldsymbol{x}) = 4xy, \text{ if } (x, y) \in [0, 1]^2, \quad f(\boldsymbol{x}) = 0, \text{ otherwise?}$$