Continuous random variables

1. The length of a baby's afternoon sleep (i hours) is a random variable X with a uniform distribution on the interval $[0, 3]$, meaning that it has the density

$$f(x) = \begin{cases} \frac{1}{3} & x \in [0,3], \\ 0 & \text{else.} \end{cases}$$

   (a) Compute the distribution function F of the random variable X and draw its graph.
   (b) What is the probability that the baby will sleep for exactly one hour? What is the probability that it sleeps for at least one hour? What is the probability that the length of its sleep will be between 30 minutes and two hours?
   (c) What is the expected mean length of the baby's sleep?
   (d) Compute the variance of the baby's sleep length.
   (e) We know that the baby is already sleeping for one hour. What is the probability that the length of the whole sleep will be longer than two hours?

2. The length of a phone call (in minutes) of a helpdesk operator is a random variable with the density

$$f(x) = \begin{cases} ce^{-x/5}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

   (a) Compute the constant $c > 0$, so that tak f truly forms a density of a random variable.
   (b) What is the mean length of a call?
   (c) Find the distribution function F and draw its graph.
   (d) What is the probability, that the phone call will be longer than 10 minutes?
   (e) The call lasted already for 5 minutes. What is the probability that it will be altogether longer than 10 minutes?
   (f) Find the variance of X.

3. (Cauchy distribution) Consider a random variable with a distribution function

$$f(x) = c \cdot \frac{1}{1+x^2}.$$

   (a) Find the constant $c > 0$, so that f truly forms a random variable.
   (b) Find the expectation EX.

Review

Continuous random variables:

Suppose that for the random variable X with a distribution function F exists such a function $f \geq 0$ such that

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Then we say that X has a **continuous distribution**. The function f is then called the **density of X.**

Properties:

- A continuous random variable can have **uncountably many** values from a subinterval of $\mathbb{R}$.
- The distribution of X is characterised by the density $f \geq 0$. For each $B \in \mathcal{B}$ then

$$\mathsf{P}(X \in B) = \int_B f(x)dx.$$

  Specially:

  (a) $1 = \mathsf{P}(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(x)dx.$
  (b) The distribution function F is continuous on $\mathbb{R}$ and can be computed as $F(x) = \mathsf{P}(X \leq x) = \int_{-\infty}^{x} f(t)dt,$
  (c) For any $a \in \mathbb{R}$ is $\mathsf{P}(X = a) = \int_{\{a\}} f(t)dt = 0.$
  (d) If $a < b$, then

$$\mathsf{P}(a < X < b) = \mathsf{P}(a < X \leq b) = F(b) - F(a) = \int_a^b f(t)dt.$$

- **The expectation (mean value)** of X is computed as

$$\mathsf{E}X = \int_{-\infty}^{\infty} xf(x)dx \quad \text{(if exists).}$$

  The expactation of $Y = h(X)$ is computed as

$$\mathsf{E}h(X) = \int_{-\infty}^{\infty} h(x)f(x)dx$$

  Specially

$$\mathsf{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx, \quad \mathsf{Var}\, X = \mathsf{E}X^2 - (\mathsf{E}X)^2.$$

- For some computations it is useful to employ the **gamma function**, which is for $p > 0$ defined as

$$\Gamma(p) = \int_0^{\infty} x^{p-1}e^{-x}dx,$$

  for which $\Gamma(p+1) = p\Gamma(p)$. If $n \in \mathbb{N}$, then $\Gamma(n) = (n-1)!$.