Bayes' theorem, discrete random variables

Bayes' theorem

1. A factory producing microchips has found out that only 87% of the produced chips have acceptable quality, the others are faulty. A quality control test has been implemented, but it is not accurate. Only 80% of the good chips pass the test and only 85% of the faulty chips actually fail the test. Only chips that have passed the test are put on market. What is the probability that a chip that passed the test is actually good?

2. A certain course at school was so hard and convoluted, that only 5% of the students understood it. The final exam of the subject was somewhat luck-based, meaning that a student with good understanding passed the exam with 95% probability, and a student who did not understand passed with 25% probability. Suppose we meet a student who passed the test. What is the probability that he actually understood the subject, given that he passed the test?

3. (Polya urn scheme) An urn contains a white and b black balls. We draw randomly one ball, note its color, put it back and add d balls of the same color into the box. Then we draw again. What is the probability of picking a white ball on the second draw?

Discrete random variables

4. We have two 500 CZK banknotes, one 1000 CZK and one 2000 CZK in our pocket. A pickpocket reaches into the pocket and steals two banknotes at random. Let X be the random variable denoting the total value of our lost money.
   (a) Find the distribution of X.
   (b) Calculate the expected (mean) loss.
   (c) Calculate the variance of X.
   (d) Plot a distribution function of X.

5. A test is composed of n questions with four possible answers a, b, c, d. For each question there is exactly one correct answer. Suppose that the student did not prepare for the test and picks the answers at random. Denote X the number of correctly answered questions.
   (a) What is the distribution of X? How is it called?
   (b) What is the mean (expected) number of correctly answered questions (EX)?
   (c) What is the variance $\mathsf{Var}\, X$?

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Bayes' theorem:

Let $A, B_1, B_2, \dots$ be random events such that $B_i \cap B_j = \emptyset \; \forall \; i \neq j$, $\bigcup_i B_i = \Omega$, $\mathsf{P}(B_i) > 0 \; \forall \; i$ and let $\mathsf{P}(A) > 0$. Then

$$\mathsf{P}(B_i|A) = \frac{\mathsf{P}(B_i \cap A)}{\mathsf{P}(A)} = \frac{\mathsf{P}(A|B_i)\mathsf{P}(B_i)}{\sum_j \mathsf{P}(A|B_j)\mathsf{P}(B_j)}.$$

Independence:

The random events A, B are independent, if

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) \cdot \mathsf{P}(B).$$

The random events $A_1, \dots A_n$ are independent, if for all $k \leq n$ and each subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$:

$$\mathsf{P}(A_{i_1} \cap \dots \cap A_{i_k}) = \mathsf{P}(A_{i_1}) \cdot \ldots \mathsf{P}(A_{i_k}).$$

(Meaning that the multiplicative property has to be verified for all pairs, triples, ... etc.)

Random variables:

A **random variable** X is a measurable mapping (function) from the space $(\Omega, \mathcal{A})$ into $(\mathbb{R}, \mathcal{B})$. To elements $\omega \in \Omega$ we assign real numbers $X(\omega)$.

- The **Distribution** of the random variable X
- describes the probabilities $\mathsf{P}(X \in B) = \mathsf{P}(\{\omega : X(\omega) \in B\})$ for all sets $B \in \mathcal{B}$,
- and is uniquely identified by the **distribution function** defined as

$$F(x) = \mathsf{P}(X \leq x) \qquad\qquad x \in \mathbb{R}.$$

Basic characteristics of the random variable X are

- **The mean value (expactation)** $\mathsf{E}X$, giving the mean (expected) value of X,
- **Variance** $\mathsf{Var}\, X$, giving the dispersion (variability) of X around its mean $\mathsf{E}X$. The variance is defined as

$$\mathsf{Var}\, X = \mathsf{E}(X - \mathsf{E}X)^2 = \mathsf{E}X^2 - (\mathsf{E}X)^2$$

  and is always **non-negative**.

Discrete random variables: If the random variable X can have only **countable many** values $x_1, x_2, \dots$, we say, that it has a **discrete distribution**.

- The distribution of X is characterised by the probabilities $p_k = \mathsf{P}(X = x_k)$, $k = 1, 2, \dots$ for which $\sum_k p_k = 1$.
- **The distribution function** is partwise constant, with jumps of the size p_k at the points x_k.
- **The expectation** EX can be computed as

$$\mathsf{E}X = \sum_k x_k \mathsf{P}(X = x_k) = \sum_k x_k p_k \quad \text{if exists.}$$

- **The expectation** of a function of the random variable $Eh(X)$ can be computed as

$$\mathsf{E}h(X) = \sum_k h(x_k)\mathsf{P}(X = x_k) = \sum_k h(x_k) p_k \quad \text{if exists.}$$