Discrete random variables

1. On average there are two faulty pixels among every 10 million produced on LCD screens. What is the probability, that on a 1280x1024 pixel screen there will be at least one faulty pixel?

2. Let X be the expected number of calls incoming to a police station in one hour. Suppose that k calls come with the probability of $\frac{\lambda^k}{k!}e^{-\lambda}$ for $k = 0, 1, 2, \ldots,$ with $\lambda > 0$.
   (a) Verify that the proposed distribution is indeed a probability distribution. What is it called?
   (b) Compute the expected number of incoming calls in one hour and its variance.

3. There are 8 keys on a ring. We are trying to open a door, for which only one key on the ring works. Because it's dark, we are selecting the keys at random. After each unsuccesfull try, the key ring falls on the floor, we pick it up and try again until we unlock the door.
   (a) What is the probability distribution of the number of unsuccessfull attempts before unlocking the door?
   (b) What is the probability of needing at most 6 unsuccessfull attempts?
   (c) What is the probability of needing at least 10 unsuccessfull attempts total, given that we already tried to open the door 6 times?
   (d) What is the expected number of unsuccessfull attempts before unlocking the door?

4. Consider a lottery where a lottery ticket is a winning one with a probability p and a losing one with $1 - p$. We devised a strategy of buying the tickets until we win.
   (a) Determine the distribution and expected number of losing tickets bought until we win.
   (b) Suppose that the winning award is $100,000$ CZK and one ticket costs 100 CZK. What is the minimum needed value of p, for our strategy to pay off?

5. There are n gentlemen who left their hats at the cloak room in a theatre. After the play ends, the absent-minded cloak room keeper gives each gentleman a hat selected at random. What is the expected number of gentlemen having their hats?

Review

Discrete random variable:

When a random variable X takes with a positive probability only **finite or countable many** values $x_1, x_2, \dots$, we say, that it has a discrete distribution.

- The probability dsitribution of X is characterised by probabilities $p_k = \mathsf{P}(X = x_k)$, $k = 1, 2, \dots$ for which necessary $\sum_k p_k = 1$.
- The **distribution function** is partwise constant, right continuous and has jumps of p_k at the points x_k.
- The **expectation or mean value** of X is computed as

$$\mathsf{E}X = \sum_k x_k \mathsf{P}(X = x_k) = \sum_k x_k p_k \quad \text{(if exists)}.$$

- **The variance** of X is computed as

$$\mathsf{Var}\, X = \mathsf{E}X^2 - (\mathsf{E}X)^2 = \sum_k x_k^2 p_k - \left(\sum_k x_k p_k\right)^2 \quad \text{(if exists)}.$$

- The expectation of the random variable $Y = h(X)$ is computed as

$$\mathsf{E}Y = \mathsf{E}h(X) = \sum_k h(x_k)\mathsf{P}(X = x_k) = \sum_k h(x_k) p_k \quad \text{(if exists)},$$

  or directly out of its distribution $\mathsf{E}Y = \sum_y y\mathsf{P}(Y = y)$.

Useful properties

- When $a, b \in \mathbb{R}$ and X is a random variable, then

$$\mathsf{E}(a + bX) = a + b\mathsf{E}X, \quad \mathsf{Var}\,(a + bX) = b^2 \mathsf{Var}\, X.$$

- When $a, b \in \mathbb{R}$ and X and Y are random variables, then

$$\mathsf{E}(aX + bY) = a\mathsf{E}X + b\mathsf{E}Y.$$