14. cvičení z MA 4 - Fourierovy řady (2/3)

Příklady

11 a) Které koeficienty Fourierovy řady funkce $f \in L^1((-\pi,\pi))$ jsou nulové, jestliže platí $f(x+\pi) = -f(x)$?

Řešení

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\pi)\,dx = \left[\begin{array}{ll} t = x+\pi & \begin{array}{c|c|c} x & -\pi & \pi \\ \hline t & 0 & 2\pi \end{array} \\ dt = dx & \end{array}\right] = -\frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = -a_0 \Rightarrow a_0 = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{-1}{\pi}\int_{-\pi}^{\pi} f(x+\pi)\cos(kx)\,dx = -\frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt-k\pi)\,dt =$$

$$= -\frac{1}{\pi}\int_0^{2\pi} f(t)\Big(\cos(kt)\underbrace{\cos(k\pi)}_{(-1)^k} + \sin(kt)\cdot\underbrace{\sin(k\pi)}_{0}\Big)\,dt$$

$$= \frac{(-1)^{k+1}}{\pi}\int_0^{2\pi} f(t)\cos(kt)\,dt = (-1)^{k+1} a_k \Rightarrow \begin{array}{l} k \text{ liché: } 0a_k = 0 \quad \text{neříká nic} \\ k \text{ sudé: } 2a_k = 0 \Rightarrow a_k = 0 \end{array}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = -\frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt-k\pi)\,dt = -\frac{1}{\pi}\int_0^{2\pi} f(t)\Big(\sin(kt)\underbrace{\cos(k\pi)}_{(-1)^k} - \cos(kt)\underbrace{\sin(k\pi)}_{0}\Big)\,dt$$

$$= (-1)^{k+1}\frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)\,dt = (-1)^{k+1} b_k \Rightarrow \begin{array}{l} k \text{ liché: } 0b_k = 0 \\ k \text{ sudé: } 2b_k = 0 \Rightarrow b_k = 0 \end{array}$$

Závěr: Všechny sudé Fourierovy koeficienty jsou nulové

$a_k = 0,\ k = 0, 2, 4, \dots$

$b_k = 0,\ k = 2, 4, \dots$

Součtové vzorce

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \Rightarrow \sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \Rightarrow \cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

$\sin(x)\sin(y) = \frac{1}{2}(-\cos(x+y) + \cos(x-y))$

(pro zapamatování z $\sin(2x) = 2\sin x\cos x$, $\cos(2x) = \cos^2 x - \sin^2 x$)

Trocha teorie navíc (nebude u zkoušky)

$f \in L^2(0,2\pi)$ pokud $\int_0^{2\pi} f^2(x)\,dx < \infty$

Věta (Parsevalova rovnost)

Nechť $f \in L^2(0,2\pi)$ a a_k a b_k jsou Fourierovy koeficienty f. Pak platí

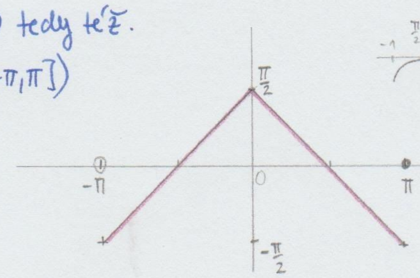
$$\int_0^{2\pi} f(x)^2\,dx = \pi\cdot\frac{a_0^2}{2} + \pi\cdot\sum_{k=1}^{\infty}(a_k^2 + b_k^2).$$

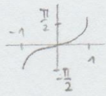
Příklad 6.1

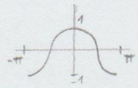
Rozviňte funkci $f(x) := \arcsin\cos x$, $x \in (-\pi,\pi]$, do Fourierovy řady na $\mathbb{R}$.
Rozhodněte, zda řada konverguje bodově na $\mathbb{R}$, a pokud ano, určete její součet.
Pomocí této řady sečtěte $\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$, $\sum_{m=0}^{\infty} \frac{1}{(2m+1)^4}$.

Řešení

$f \in P_{2\pi}$: $\cos x$ je 2π-periodická funkce, $\arcsin(\cos x)$ tedy též.
$\arcsin\cos x$ je spojitá na $[-\pi,\pi] \Rightarrow L^1([-\pi,\pi])$







f je sudá, neboť $\cos x$ je sudá

$b_k = 0$, $k \in \mathbb{N}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \arcsin(\cos x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)dx = \frac{2}{\pi}\left[-\frac{x^2}{2}+\frac{\pi}{2}x\right]_0^{\pi} = \frac{2}{\pi}\left(-\frac{\pi^2}{2}+\frac{\pi^2}{2}\right) = 0$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} \arcsin(\cos x)\cos kx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)\cos kx\,dx = \int_0^{\pi}\cos kx\,dx - \frac{2}{\pi}\int_0^{\pi} \underset{\downarrow\,1}{x}\,\underset{\uparrow\,\frac{\sin kx}{k}}{\cos kx}\,dx$$

$$= 0 + -\frac{2}{\pi}\left[x\frac{\sin kx}{k}\right]_0^{\pi} + \frac{2}{\pi k}\int_0^{\pi}\sin kx\,dx = \frac{2}{\pi k}\left[-\frac{\cos kx}{k}\right]_0^{\pi} = \frac{2}{k^2\pi}\left((-1)^{k+1}+1\right) = \begin{cases} 0, & k \text{ sudé} \\ \frac{4}{k^2\pi}, & k \text{ liché} \end{cases}$$

Fourierova řada funkce f: $\underline{\underline{\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos((2m+1)x)}}$

Funkce $f \in P_{2\pi}$ je spojitá na $\mathbb{R}$ a je BV na $[-\pi,\pi]$, neboť je zde po částech C^1.
Z J-D kritéria tedy Fourierova řada konverguje bodově k funkci f.
Navíc $S_n^f \overset{loc}{\rightrightarrows} f$ na $(-2\pi,2\pi)$, tedy $S_m^f \rightrightarrows f$ na $[-\pi,\pi]$ a tedy $S_m^f \rightrightarrows f$ na $\mathbb{R}$.

$$f(x) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos((2m+1)x)$$

Součty

$$\frac{\pi}{2} = f(0) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \Rightarrow \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \underline{\underline{\frac{\pi^2}{8}}}$$

Z Parsevalovy rovnosti máme ($f \in L^2$ ze spojitosti na $[-\pi,\pi]$)

$$\pi\cdot\sum_{m=0}^{\infty}\overset{= a_k^2}{\frac{16}{\pi^2(2m+1)^4}} = \int_{-\pi}^{\pi} f(x)^2\,dx = 2\cdot\int_0^{\pi} f(x)^2\,dx = 2\cdot\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)^2dx = 2\cdot\int_0^{\pi}\left(x^2-\pi x+\frac{\pi^2}{4}\right)dx = \frac{\pi^3}{6}$$

Tedy

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \underline{\underline{\frac{\pi^4}{96}}}$$

Teoretický podklad (pro jednoduchost na $\mathbb{C}$)

$s_n^f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt}$	$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$ Dirichletovo jádro	$s_n^f = f * D_n$
$\sigma_n^f(t) = \frac{1}{n+1} \sum_{j=0}^{n} s_j^f(t)$ "Fejérovské součty"	$K_n(t) = \frac{1}{n+1} \sum_{j=0}^{n} D_j(t)$ Fejérovo jádro	$\sigma_n^f = f * K_n$

↳ Cesàrovský součet. $\sum a_n = \sigma \Rightarrow (C)\sum a_n = \sigma$ (víme-li, že $\sum a_n$ k a jak vypadá součet $(C)\sum a_n$ $\Rightarrow$ víme i, jak vypadá součet $\sum a_k$)

Fejérova věta (Věta 23.5)

Nechť $f \in P_{2\pi}$ a $t \in \mathbb{R}$. Jestliže $\exists$ vlastní limity $f(t+)$ a $f(t-)$, pak $\lim_{n\to\infty} \sigma_n^f = \frac{f(t+)+f(t-)}{2}$ bodová

Jestliže $a,b \in \mathbb{R}^*$, $a<b$, a f je spojitá na (a,b), pak $\sigma_n^f \overset{loc}{\rightrightarrows} f$ na (a,b). stejnoměrná

Věta 23.7

Nechť $f \in P_{2\pi}$. Pak $\|f - \sigma_n^f\|_{L^1} \to 0$ v L^1 normě

Příklad 6.12

Nechť funkce f je definována na $\mathbb{R}$ předpisem $f(x) = \operatorname{sgn}(\sin(2x)) + \cos^3 x + \sin^3 x$. Spočtěte Fourierovu řadu funkce f a určete součet její Fejérovy řady pro každé $x \in \mathbb{R}$.

Řešení

f je 2π-periodická: $\left.\begin{array}{l} \operatorname{sgn}(\sin(2x)) \ \checkmark \\ \cos^3(x) \ \checkmark \\ \sin^3(x) \ \checkmark \end{array}\right\}$ součet 2π-periodických je 2π-periodický

f je $L^1([-\pi,\pi])$ neboť je omezená a měřitelná

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\underbrace{\operatorname{sgn}(\sin 2x)}_{\text{liché}} + \underbrace{\cos^3 x}_{\int_{-\pi}^{\pi}\cos^3 x\,dx = 0} + \underbrace{\sin^3 x}_{\text{liché}}\Big)dx = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\underbrace{\operatorname{sgn}(\sin 2x)}_{\text{liché}} + \underbrace{\cos^3 x}_{\text{sudé}} + \underbrace{\sin^3 x}_{\text{liché}}\Big)\underbrace{\cos kx}_{\text{sudé}}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^3 x \cos kx\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(\operatorname{sgn}(\sin 2x) + \cos^3 x + \sin^3 x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^3 x \sin kx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{sgn}(\sin 2x)\sin kx\,dx$$

- $$\int_{-\pi}^{\pi}\cos^3 x\cos kx\,dx = \int_{-\pi}^{\pi}(1-\sin^2 x)\cos x\cos kx\,dx = \int_{-\pi}^{\pi}\cos x\cos kx\,dx - \int_{-\pi}^{\pi}\sin^2 x\cos x\cos kx\,dx =$$

$$= \int_{-\pi}^{\pi}\cos x\cos kx\,dx - \int_{-\pi}^{\pi}(\cos^2 x - \cos 2x)\cos x\cos kx\,dx = \int_{-\pi}^{\pi}\cos x\cos kx\,dx - \int_{-\pi}^{\pi}\cos^3 x\cos kx + \int_{-\pi}^{\pi}\cos 2x\cos x\cos kx\,dx$$

$$2\int_{-\pi}^{\pi}\cos^3x\cos kx\,dx=\int_{-\pi}^{\pi}\cos x\cos kx\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos kx\,(\cos 3x+\cos x)\,dx=\frac{3}{2}\int_{-\pi}^{\pi}\cos x\cos kx\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos 3x\cos kx\,dx$$

$$\int_{-\pi}^{\pi}\cos^3x\cos kx\,dx=\begin{cases}\frac{3}{4}\pi &, k=1\\ \frac{1}{4}\pi &, k=3\\ 0 &, k\in\mathbb{N}\setminus\{1,3\}.\end{cases}\qquad\Rightarrow\qquad a_k=\begin{cases}\frac{3}{4} &, k=1\\ \frac{1}{4} &, k=3\\ 0 &, k\in\mathbb{N}\setminus\{1,3\}\end{cases}$$

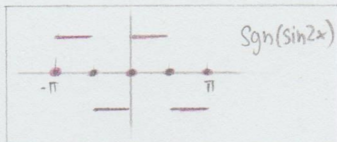
- $$\int_{-\pi}^{\pi}\underbrace{\sin^3x}_{=(1-\cos^2x)\sin x}\sin kx\,dx=\int_{-\pi}^{\pi}\sin x\sin kx-\int_{-\pi}^{\pi}(\cos 2x+\sin^2x)\sin x\sin kx\,dx$$

$\cos 2x\sin x=\frac{1}{2}(\sin 3x+\underbrace{\sin(-x)}_{=-\sin(x)})$

$$2\int_{-\pi}^{\pi}\sin^3x\sin kx\,dx=\int_{-\pi}^{\pi}\sin x\sin kx-\frac{1}{2}\int_{-\pi}^{\pi}\sin 3x\sin kx\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\sin x\sin kx\,dx$$

$$\int_{-\pi}^{\pi}\sin^3x\sin kx\,dx=\begin{cases}\frac{3}{4}\pi &, k=1\\ -\frac{1}{4}\pi &, k=3\\ 0 &, k\in\mathbb{N}\setminus\{1,3\}\end{cases}$$

- $$\int_{-\pi}^{\pi}\underbrace{\operatorname{sgn}(\sin 2x)\sin kx}_{\text{lichá}}\,dx=2\cdot\int_{0}^{\pi}\operatorname{sgn}(\sin 2x)\sin kx\,dx=$$



$$=2\left(\int_0^{\frac{\pi}{2}}\sin kx\,dx-\int_{\frac{\pi}{2}}^{\pi}\sin kx\,dx\right)=2\left(\left[-\frac{\cos kx}{k}\right]_0^{\frac{\pi}{2}}-\left[-\frac{\cos k}{k}\right]_{\frac{\pi}{2}}^{\pi}\right)=2\left(-\frac{\cos k\frac{\pi}{2}}{k}+\frac{1}{k}+\frac{(-1)^k}{k}-\frac{\cos k\frac{\pi}{2}}{k}\right)=$$

$$=\frac{2}{k}\left(-2\cos k\frac{\pi}{2}+1+(-1)^k\right)=\begin{cases}-\frac{4}{k}\cos k\frac{\pi}{2}=0 &, k\text{ liché}\\ \frac{4}{k}\left(-\cos k\frac{\pi}{2}+1\right) &, k\text{ sudé}\end{cases}$$

$$\frac{4}{2m}\left(-\cos\left(2m\frac{\pi}{2}\right)+1\right)=\frac{2}{m}\left((-1)^{m+1}+1\right)\begin{cases}0 &, m\text{ sudé}\\ \frac{4}{m} &, m\text{ liché}\end{cases}\qquad=\frac{8}{k}$$

$\Rightarrow$ nenulové jsou pouze členy s $k=2(2l+1)=4l+2$, $l=0,1,\dots$

$$b_k=\begin{cases}\frac{3}{4} &, k=1\\ -\frac{1}{4} &, k=3\\ \frac{8}{\pi k} &, k=4l+2,\ l=0,1,\dots\\ 0 &, k\in\mathbb{N}\setminus\{1,3,4l+2;\ l\in\mathbb{N}_0\}\end{cases}$$

Fourierova řada funkce f je tedy

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx=0+\frac{3}{4}\cos x+\frac{3}{4}\sin x+\frac{1}{4}\cos 3x-\frac{1}{4}\sin 3x+\sum_{m=0}^{\infty}\frac{8}{\pi(4m+2)}\sin((4m+2)x).$$

$f\in P_{2\pi}$, tedy splňuje předpoklad Fejérovy věty a $\lim\sigma_m^f=\frac{1}{2}(f(t+)+f(t-))$

Navíc $\forall t\in\mathbb{R}$: $f(t)=\frac{1}{2}(f(t+)+f(t-))$, tedy $\lim_{n\to\infty}\sigma_m^f(t)=f(t)\ \ \forall t\in\mathbb{R}$.

$f(0-)=-1+1+0=0$

$f(0+)=1+1+0=2$ apod.

$f(0)=0+1+0=1$