Approximation properties of the classes of flat modules originating from algebraic geometry

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Jan Trlifaj (Univerzita Karlova, Praha)

Motivation - the Quillen-Hovey theory

[Quillen'1967]

Let $\mathcal{G}$ be a Grothendieck category. The unbounded derived category $D(\mathcal{G})$ can be studied via model category structures on the category $\mathcal{C}(\mathcal{G})$ of unbounded chain complexes over $\mathcal{G}$:

Morphisms between A and B of $D(\mathcal{G})$ are the $\mathcal{C}(\mathcal{G})$-morphisms between cofibrant and fibrant replacements of A and B, respectively, modulo chain homotopy.

[Hovey'2002]

Such model category structures correspond to functorially complete cotorsion pairs in $\mathcal{C}(\mathcal{G})$.

A basic example from algebraic geometry

$\mathcal{G} = Qcoh(X)$, the category of quasi-coherent sheaves on a scheme X.

Quasi-coherent sheaves as representations

Let X be a scheme and $\mathcal{O}_X$ its structure sheaf.

[Enochs-Estrada'2005]

A quasi-coherent sheaf Q on X can be represented by an assignment

- to every affine open subscheme $U \subseteq X$, an $\mathcal{O}_X(U)$-module $Q(U)$ of sections, and
- to each pair of embedded affine open subschemes $V \subseteq U \subseteq X$, an $\mathcal{O}_X(U)$-homomorphism $f_{UV} : Q(U) \to Q(V)$ such that

$$\mathrm{id}_{\mathcal{O}_X(V)} \otimes f_{UV} : \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} Q(U) \to \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} Q(V) \cong Q(V)$$

is an $\mathcal{O}_X(V)$-isomorphism.

$+$ compatibility conditions for the f_{UV}.

Properties of the representations

Exactness

The functors $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} -$ are exact, i.e., the $\mathcal{O}_X(U)$-modules $\mathcal{O}_X(V)$ are flat.

The affine case

If $X = \mathrm{Spec}(R)$ for a commutative ring R, then $Qcoh(X) \cong \text{Mod-}R$.

Non-uniqueness of the representations

Not all affine open subschemes are needed: a set of them, $\mathcal{S}$, covering both X, and all $U \cap V$ where $U, V \in \mathcal{S}$, will do.

Extending properties of modules to quasi-coherent sheaves

Examples

If each module of sections is

- projective,
- (restricted) flat Mittag-Leffler,
- flat,

then the quasi-coherent sheaf Q is called

- an infinite dimensional vector bundle,
- (restricted) Drinfeld vector bundle,
- flat quasi-coherent sheaf.

[Raynaud-Gruson'1971, Estrada-Guil-T.'2014]

The notions above are local, i.e., independent of the representation (choice of the affine open covering $\mathcal{S}$ of the scheme X).

Flat Mittag-Leffler modules

[Raynaud-Gruson'1971]

A module M is flat Mittag-Leffler provided that each finite subset of M is contained in a countably generated projective and pure submodule of M. Notation: $\mathcal{FM}$.

[Herbera-T.'2012]

Equivalently: M is locally $\mathcal{C}$-free, where $\mathcal{C}$ is the class of all countably presented projective modules.

A basic definition

Let $\mathcal{C}$ be a class of countably presented modules.

A module M is locally $\mathcal{C}$-free provided there exists a set $\mathcal{S} \subseteq \mathcal{C}$ consisting of submodules of M such that

- each countable subset of M is contained in a module from $\mathcal{S}$, and
- $\mathcal{S}$ is closed under unions of countable chains.

Cohomology of quasi-coherent sheaves

[Gillespie'2004, Estrada-Guil-Prest-T.'2012]

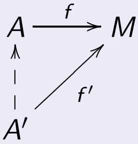
Introduce a method of constructing functorially complete cotorsion pairs in $\mathcal{C}(Qcoh(X))$, and hence model category structures, using complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ of modules such that $\mathcal{A} \subseteq \mathcal{F}_0$.

A pair of classes $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in Mod-R if

- $\mathcal{A} = {}^{\perp}\mathcal{B} := \{A \in \text{Mod-}R \mid \text{Ext}^i_R(A, B) = 0 \text{ for all } i \geq 1 \text{ and } B \in \mathcal{B}\}$,
- $\mathcal{B} = \mathcal{A}^{\perp}$,
- for each module M there is an exact sequence $0 \to B \to A \to M \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (i.e., $\mathcal{A}$ is a special precovering class), and
- for each module M' there is an exact sequence $0 \to M' \to B' \to A' \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ (i.e., $\mathcal{B}$ is a special preenveloping class).

Module approximations

A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ factorizes through f:



f is an $\mathcal{A}$-precover of M. If f is also right minimal (i.e., f factorizes through itself only by an automorphism of A), then f is an $\mathcal{A}$-cover of M. If f is surjective and $\mathrm{Ext}^1_R(\mathcal{A}, \mathrm{Ker}(f)) = 0$, then f is called special. If $\mathcal{A}$ provides for covers (special precovers) of all modules, then $\mathcal{A}$ is called a covering (special precovering) class.

Preenveloping and (special) enveloping classes are defined dually.

Transfinite extensions

Let $\mathcal{C}$ be a class of modules. A module M is $\mathcal{C}$-filtered (or a transfinite extension of the modules in $\mathcal{C}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$.

Notation: $\mathrm{Trans}(\mathcal{C})$.

Example

Let R be a ring and $\mathcal{C}$ the class of all simple modules. Then $\mathrm{Trans}(\mathcal{C})$ is the class of all semiartinian modules.

The abundance of approximations

[Eklof-T.'2000]

For each set of modules $\mathcal{S}$, there is a complete cotorsion pair $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$.

[Enochs'2012, Šťovíček'2012]

The class $\mathrm{Trans}(\mathcal{S})$ is precovering for each set of modules $\mathcal{S}$.

Some examples

[Enochs et al.]

- For each ring R and each $n \geq 0$, the class $\mathcal{P}_n$ is special precovering, $\mathcal{F}_n$ is covering, and $\mathcal{I}_n$ is special preenveloping.
- For each Iwanaga-Gorenstein ring R, the class $\mathcal{GP}$ is special precovering, and $\mathcal{GI}$ is special preenveloping.
- ...

Does $\mathcal{FM}$ fit in this context?

Flat Mittag-Leffler approximations

Theorem (Angeleri-Šaroch-T.)

$\mathcal{FM}$ is (pre) covering, iff R is a right perfect ring (i.e., $\mathcal{P}_0 = \mathcal{F}_0$).

What is the obstruction for existence of precovers?

Bass modules

Let R be a ring, and $\mathcal{C}$ be a class of countably presented modules.

$\varinjlim_{\omega} \mathcal{C}$ denotes the class of all Bass modules over $\mathcal{C}$, that is, the modules B that are countable direct limits of modules from $\mathcal{C}$. W.l.o.g., such B is the direct limit of a chain

$$C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} C_i \xrightarrow{f_i} C_{i+1} \xrightarrow{f_{i+1}} \dots$$

with $C_i \in \mathcal{C}$ and $f_i \in \mathrm{Hom}_R(C_i, C_{i+1})$ for all $i < \omega$.

Classic Bass modules

Let $\mathcal{C}$ be the class of all finitely generated projective modules. Then the Bass modules coincide with the countably presented flat modules.

If R is not right perfect, then an instance of such a classic Bass module B arises when $F_i = R$ and f_i is the left multiplication by a_i ($i < \omega$). Note: B is not projective, hence not flat Mittag-Leffler.

Proof of the theorem

It suffices to prove that if R is not right perfect, then the class $\mathcal{FM}$ is not precovering.

Let B be a non-projective classic Bass module. Assume there exists a $\mathcal{FM}$-precover $f : F \to B$. Let $K = \mathrm{Ker}(f)$, so we have an exact sequence

$$0 \to K \hookrightarrow F \xrightarrow{f} B \to 0.$$

Let κ be an infinite cardinal such that $|R| \leq \kappa$ and $|K| \leq 2^\kappa = \kappa^\omega$. Then there exists a 'tree-module' short exact sequence

$$0 \to D \hookrightarrow G \to B^{(2^\kappa)} \to 0$$

such that $G \in \mathcal{FM}$ and D is a free module of rank κ. Clearly, $G \in \mathcal{P}_1$.

Let $\eta : K \to E$ be a $\{G\}^{\perp}$-preenvelope of K with a $\{G\}$-filtered cokernel. Consider the pushout

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{\subseteq} & F & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \eta\downarrow & & \varepsilon\downarrow & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{\subseteq} & P & \xrightarrow{g} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Coker}(\eta) & \xrightarrow{\cong} & \mathrm{Coker}(\varepsilon) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Then $P \in \mathcal{FM}$. Since f is an $\mathcal{FM}$-precover, there exists $h : P \to F$ such that $fh = g$. Then $f = g\varepsilon = fh\varepsilon$, whence $K + \mathrm{Im}(h) = F$. Let $h' = h \restriction E$. Then $h' : E \to K$ and $\mathrm{Im}(h') = K \cap \mathrm{Im}(h)$.

Consider the 'restricted' exact sequence

$$0 \longrightarrow \mathrm{Im}(h') \xrightarrow{\subseteq} \mathrm{Im}(h) \xrightarrow{f \restriction \mathrm{Im}(h)} B \longrightarrow 0.$$

As $E \in G^{\perp}$ and $G \in \mathcal{P}_1$, also $\mathrm{Im}(h') \in G^{\perp}$. Applying $\mathrm{Hom}_R(-, \mathrm{Im}(h'))$ to the 'tree-module' exact sequence above, we obtain the exact sequence

$$\mathrm{Hom}_R(D, \mathrm{Im}(h')) \to \mathrm{Ext}^1_R(B, \mathrm{Im}(h'))^{2^{\kappa}} \to 0$$

where the first term has cardinality $\leq |K|^{\kappa} \leq 2^{\kappa}$, so the second term must be zero.

This yields $\mathrm{Im}(h') \in B^{\perp}$. Then $f \restriction \mathrm{Im}(h)$ splits, and so does the $\mathcal{FM}$-precover f, a contradiction with $B \notin \mathcal{FM}$. □

The general case

Šaroch's lemma on Bass modules

Let $\mathcal{C}$ be a class of countably presented modules, and $\mathcal{L}$ the class of all locally $\mathcal{C}$-free modules.

Let B be a Bass module over $\mathcal{C}$ such that B is not a direct summand in a module from $\mathcal{L}$.

Then **the Bass module B has no $\mathcal{L}$-precover**.

A connection to tilting theory

T is a (large) tilting module provided that

- $\mathrm{pd}(T) < \infty$,
- $\mathrm{Ext}^i_R(T, T^{(X)}) = 0$ for each $i \geq 1$ and each set X,
- There exists $r < \omega$ and an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ with $T_j \in \mathrm{Add}(T)$ for all $j \leq r$.

$\mathcal{T}_T := T^\perp$ is the right tilting class, and

$\mathcal{A}_T = {}^\perp(T^\perp)$ is the left tilting class induced by T.

The tilting generalization

Replacements

The projective module $R \leftrightarrow$ any tilting module T,

$\mathcal{P}_0 \leftrightarrow$ the left tilting class $\mathcal{A}_T$,

$\mathcal{F}_0 \leftrightarrow$ the direct limit closure $\varinjlim \mathcal{A}_T$,

$\mathcal{FM} \leftrightarrow$ the class $\mathcal{L}$ of all locally T-free modules, i.e., the locally $\mathcal{C}$-free modules, where $\mathcal{C}$ is the class of all countably presented modules from $\mathcal{A}_T$.

Theorem (Angeleri-Šaroch-T.)

$\mathcal{L}$ is (pre) covering, iff $\mathcal{A}_T = \varinjlim \mathcal{A}_T$, iff each pure embedding in Add(T) splits (i.e., T is $\sum$-pure split).

The dual setting: contraherent cosheaves

Definition (Positselski)

Let X be a scheme and $\mathcal{O}_X$ its structure sheaf.
A contraherent cosheaf P on X can be represented by an assignment

- to every affine open subscheme $U \subseteq X$, of an $\mathcal{O}_X(U)$-module $P(U)$ of cosections, and
- to each pair of embedded affine open subschemes $V \subseteq U \subseteq X$, an $\mathcal{O}_X(U)$-homomorphism $g_{VU} : P(V) \to P(U)$ such that

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), g_{VU}) : P(V) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), P(U))$$

is an $\mathcal{O}_X(V)$-isomorphism.

$+$ compatibility conditions for the g_{VU}.

A drawback and a remedy

The drawback

The $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is only flat, but not projective in general, so the Hom-functor above is not exact.

The remedy

Exactness is forced by an extra condition on the contraherent cosheaf P:

$$\mathrm{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), P(U)) = 0.$$

Moreover, the $\mathcal{O}_X(U)$-modules $\mathcal{O}_X(V)$ are very flat ...

Very flat modules

Definition

A module M over a commutative ring R is very flat provided that $M \in {}^{\perp}(\mathcal{S}^{\perp})$ where $\mathcal{S} = \{R[s^{-1}] \mid s \in R\}$ and $R[s^{-1}]$ denotes the localization of R at the multiplicative set $\{1, s, s^2, \dots\}$.
Notation: $\mathcal{VF} := {}^{\perp}(\mathcal{S}^{\perp})$.

Lemma (Positselski)

Let $R \to S$ be a homomorphism of commutative rings such that the induced morphism of affine schemes $Spec(S) \to Spec(R)$ is an open embedding. Then S is a very flat R-module.

Basic properties of very flat modules

- $\mathcal{P}_0 \subseteq \mathcal{VF} \subseteq \mathcal{F}_0 \cap \mathcal{P}_1$.
- There is a complete cotorsion pair $(\mathcal{VF}, \mathcal{CA})$. The modules in $\mathcal{CA}$ are called contraadjusted.
- $\mathcal{VF} = \mathrm{Trans}(\mathcal{VF}^{\leq \omega})$.

Locally very flat modules

Definition

A module M is locally very flat provided there exists a set $\mathcal{E}$ consisting of countably presented very flat submodules of M such that each countable subset of M is contained in an element of $\mathcal{E}$, and $\mathcal{E}$ is closed under unions of countable chains.

Notation: $\mathcal{LV}$.

Basic properties

Since $\mathcal{P}_0 \subseteq \mathcal{VF}$, we have $\mathcal{FM} \subseteq \mathcal{LV} \subseteq \mathcal{F}_0$.

Also $\mathcal{EC} \subseteq \mathcal{CA}$. If R is a domain, then $\mathcal{DI} \subseteq \mathcal{CA}$.

Example: the case of a Dedekind domain

Lemma (Slávik-T.)

Let R be a Dedekind domain and M be a module.

- $\mathcal{VF} = \mathrm{Trans}(\mathcal{T})$, *where $\mathcal{T}$ is the set of all submodules of the modules in $\mathcal{S}$.*
- *If M is of finite rank, then $M \in \mathcal{VF}$, iff there exists $0 \neq s \in R$ such that $M \otimes_R R[s^{-1}]$ is a projective $R[s^{-1}]$-module.*
- *('Pontryagin Criterion') $M \in \mathcal{LV}$, iff each finite subset of M is contained in a countably generated very flat pure submodule of M, iff each finite rank submodule of M is very flat.*

Locally very flat modules and precovers

Theorem (Slávik-T.)

Let R be a noetherian domain. Then the following conditions are equivalent:

- $\mathcal{LV}$ *is a (pre) covering class,*
- $\mathcal{VF}$ *is a covering class,*
- *Spec*(R) *is finite,*
- $\mathcal{VF} = \mathcal{F}_0$.

In this case, R has Krull dimension 1.

References

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