Tilting theory for commutative rings

Jan Trlifaj

Univerzita Karlova, Praha

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(i-) Tilting Modules
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1. $\text{pd}_R(T) \leq n$, i.e., there is an $\text{Add}(R)$-resolution of $T$ of length $\leq n$.
2. $\text{Ext}_R^i(T, T^{(\kappa)}) = 0$ for all $1 \leq i$ and all $\kappa$, i.e., $T$ is a strong splitter.
3. There is a long exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \text{Add} T$. 

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(T3) There is a long exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \text{Add} T$, i.e., there is an $\text{Add}(T)$-coresolution of $R$ of length $\leq n$. 

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Tilting module = $n$–tilting module for some $n < \omega$. The tilting class induced by $T$ is $T^\perp = \{ M \in \text{Mod-}R \mid \text{Ext}_R^i(T, M) = 0 \text{ for all } i \geq 1 \}$. 
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A tilting module $T$ is good if (T3) holds with $\text{Add}T$ replaced by $\text{add}T$.

The tilting modules $T$ and $T'$ are equivalent if $T^\perp = (T')^\perp$.

Each tilting module is equivalent to a good one.

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Each classical tilting module is good.

Theorem

Let $T$ be a classical $n$–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} \text{Ker}(\text{Ext}_R^i(T, -)) \quad \cong \quad \text{Ext}_R^i(T, -) \quad \Leftrightarrow \quad \bigcap_{j \leq n, j \neq i} \text{Ker}(\text{Tor}_S^j(-, T))$$

where $S = \text{End}_R(T)$.
Classical tilting for commutative rings is trivial ...
Lemma

Let $R$ be a commutative ring and $T$ be a strongly finitely presented module of projective dimension $n \geq 1$. Then $\text{Ext}^n_R(T, T) \neq 0$. 
Classical tilting for commutative rings is trivial ...

Lemma

- Let $R$ be a commutative ring and $T$ be a strongly finitely presented module of projective dimension $n \geq 1$. Then $\text{Ext}_R^n(T, T) \neq 0$.
- All classical tilting modules over a commutative ring are projective. !!!
Lemma

- Let $R$ be a commutative ring and $T$ be a strongly finitely presented module of projective dimension $n \geq 1$. Then $\text{Ext}^n_R(T, T) \neq 0$.
- All classical tilting modules over a commutative ring are projective. !!!
General $i$-tilting theorem

Let $R$ be a ring and $T$ be a good $n$–tilting module. Then for each $i \leq n$ there is a category equivalence

$$\bigcap_{j \leq n, j \neq i} \text{Ker}(\text{Ext}^j_R(T, -)) \leftrightarrow \text{Ext}^i_R(T, -) \leftrightarrow \bigcap_{j \leq n, j \neq i} \text{Ker}(\text{Tor}^S_j(-, T)) \cap \mathcal{E}_\perp$$

where $S = \text{End}_R(T)$, $\mathcal{E}_\perp = \{X \in D(S) \mid \text{Hom}_{D(S)}(\mathcal{E}, X) = 0\}$, and $\mathcal{E}$ is the kernel of the total left derived functor $L(- \otimes_S T)$. 
Tilting classes and definability
Let $R$ be a ring, $n < \omega$, and $\mathcal{T}$ be a class of modules.
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In particular, each tilting class is definable, i.e., closed under direct products, direct limits, and pure submodules.
Tilting for commutative noetherian domains: the one-dimensional case
Theorem

Let \( R \) be a commutative noetherian domain of Krull dimension 1. Then tilting classes are parametrized by the subsets of \( \text{mSpec}(R) \).
Tilting for commutative noetherian domains: the one-dimensional case

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Given a $P \subseteq \text{mSpec}(R)$, the corresponding tilting class is

$$I_P = \{ M \in \text{Mod}-R \mid M \cdot p = M \text{ for all } p \in P \}.$$
Theorem

Let $R$ be a commutative noetherian domain of Krull dimension 1. Then tilting classes are parametrized by the subsets of $m\text{Spec}(R)$.

Given a $P \subseteq m\text{Spec}(R)$, the corresponding tilting class is

$$T_P = \{ M \in \text{Mod-}R \mid M \cdot \mathfrak{p} = M \text{ for all } \mathfrak{p} \in P \}.$$

This class is induced by the **Bass tilting module**, i.e., the tilting module $T_P = R_P \oplus R_P/R$ where $R_P = \bigcap_{q \in m\text{Spec}(R) \setminus P} R_q$ and $R_q$ denotes the localization of $R$ at $q$. 
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The module \( F_P \) is a tilting module of projective dimension \( \leq 1 \), called the **Fuchs tilting module** for \( P \).
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Tilting modules over valuation domains

**Theorem**

Let $R$ be a valuation domain. The Fuchs tilting modules $\mathcal{F}_P$ where $P$ runs over all prime ideals in $R$, classify all tilting modules up to equivalence.

The corresponding tilting classes are

$$\mathcal{T}_P = \{ M \in \text{Mod}-R \mid M_s = M \text{ for all } s \in R \setminus P \}.$$
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Definition

\[ \mathcal{L}_f = \text{the set of all finitely generated ideals in a finitely generated localizing system } \mathcal{L}. \]
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Let $H$ be the submodule of $G$ generated by the elements $\{x_\sigma \mid \sigma \in \Lambda\} \in G$. 
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Define \( S_\mathcal{L} = G/H \).
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Definition

$L_f$ = the set of all finitely generated ideals in a finitely generated localizing system $L$. $\Lambda$ = the set of all finite sequences of elements of $L_0$ (including the empty sequence $\emptyset$). Let $G_\emptyset = R$ and for $\emptyset \neq \lambda = (I_0, \ldots, I_k) \in \Lambda$, define $G_\lambda = I_0^{-1} \cdots I_k^{-1} \subseteq Q$ and $\lambda^- = (I_0, \ldots, I_{k-1})$ for $k > 0$ and $\lambda^- = \emptyset$ otherwise. Let $G = \bigoplus_{\lambda \in \Lambda} G_\lambda$.

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Define $S_L = G/H$.

Then $S_L$ is a tilting module of projective dimension $\leq 1$,
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Define \( S_\mathcal{L} = G/H \).

Then \( S_\mathcal{L} \) is a tilting module of projective dimension \( \leq 1 \), called the Salce tilting module for \( \mathcal{L} \).
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The corresponding tilting classes are

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Remark: These are exactly the special preenveloping torsion classes in $\text{Mod–}R$. 
Tilting for commutative noetherian rings:
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$$\mathcal{T}_P = \{ M \in \text{Mod-}R \mid \text{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P \}.$$
Let $R$ be a 1–Gorenstein ring. Then tilting classes are parametrized by the subsets of the set $P_1$ of all prime ideals of height 1. Given $P \subseteq P_1$, the corresponding tilting class is

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T_P = \{ M \in \text{Mod-}R \mid \text{Ext}^1_R(E(R/p), M) = 0 \text{ for all } p \in P \}.
$$

This class is induced by the tilting module $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ where $R_P$ is the subring of $Q_{cl}(R)$ containing $R$ and satisfying $R_P/R \cong \bigoplus_{p \in P} E(R/p)$. 
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This class is induced by the tilting module \( T_P = R_P \oplus \bigoplus_{p \in P} E(R/p) \) where \( R_P \) is the subring of \( Q_{cl}(R) \) containing \( R \) and satisfying \( R_P/R \cong \bigoplus_{p \in P} E(R/p) \). The \( T_P \) is called the Bass tilting module.
Let $R$ be a 1–Gorenstein ring. Then tilting classes are parametrized by the subsets of the set $P_1$ of all prime ideals of height 1.

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This class is induced by the tilting module $T_P = R_P \oplus \bigoplus_{p \in P} E(R/p)$ where $R_P$ is the subring of $Q_{cl}(R)$ containing $R$ and satisfying $R_P/R \cong \bigoplus_{p \in P} E(R/p)$. The $T_P$ is called the Bass tilting module.

Moreover, $T_P = S_P^{\perp}$, where $S_P = \{ F_p \mid p \in P \}$, and $F_p$ is the Auslander–Buchweitz approximation of $R/p$. 
Tilting for regular rings of Krull dimension two
Tilting for regular rings of Krull dimension two

The representing tilting modules have been characterized only in the local case.
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1. ordinary 1–dimensional ( = generalized Fuchs tilting modules),
2. ordinary 2–dimensional (obtained by localization), and
3. two exceptional tilting modules $T_e$ and $T_f$. 
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There are of three kinds:

1. ordinary 1–dimensional (= generalized Fuchs tilting modules),
2. ordinary 2–dimensional (obtained by localization), and
3. two exceptional tilting modules $T_e$ and $T_f$.

Example

The tilting class $\mathcal{I}_1$ is induced by an exceptional tilting module $T_e$ such that $T_e$ is countably generated, torsionfree, and $\text{pd} T_e = 1$. 
The dual setting
The dual setting

**Definition**

Let $R$ be a ring and $n < \omega$. A left $R$–module $C$ is $n$–cotilting provided

(C1) $\text{id}_R(C) \leq n$.

(C2) $\text{Ext}^i_R(C^\kappa, C) = 0$ for all $1 \leq i$ and all cardinals $\kappa$.

(C3) There is an injective cogenerator $W$ and a long exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to W \to 0,$$

with $C_i \in \text{Prod}C$.

The class $\perp C = \{ M \in R\text{-Mod} \mid \text{Ext}^i_R(M, C) = 0$ for all $i \geq 1 \}$ is the cotilting class induced by $C$.

The cotilting modules $C$ and $C'$ are equivalent if $\perp C = \perp C'$.
Duality: formal versus explicit
The notions of a cotilting and tilting module are formally dual, but there is also an explicit duality:
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Let $R$ be a ring, $n \geq 0$, and $T$ be an $n$–tilting right $R$–module. Then the dual module $C = T^* = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is an $n$–cotilting left $R$–module.
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The tilting right $R$–modules $T$ and $T'$ are equivalent, iff the dual modules $T^*$ and $(T')^*$ are equivalent cotilting left $R$–modules.
The notions of a cotilting and tilting module are formally dual, but there is also an explicit duality:

Let \( R \) be a ring, \( n \geq 0 \), and \( T \) be an \( n \)–tilting right \( R \)–module. Then the dual module \( C = T^* = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z}) \) is an \( n \)–cotilting left \( R \)–module.

The tilting right \( R \)–modules \( T \) and \( T' \) are equivalent, iff the dual modules \( T^* \) and \( (T')^* \) are equivalent cotilting left \( R \)–modules.

Moreover, if \( S \) is a set consisting of strongly finitely presented modules of projective dimension \( \leq n \) such that \( T^\perp = S^\perp \) is the tilting class induced by \( T \), then

\[
\perp T^* = S^T = \{ N \in R\text{-Mod} \mid \text{Tor}_i^R(S, N) = 0 \text{ for all } i \geq 1 \text{ and } S \in S \}
\]

is the cotilting class induced by \( T^* \).
Cofinite type
Cofinite type

The cotilting modules and classes of the form $T^*$ and $\perp T^*$, respectively, are called of cofinite type.
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The map $T \mapsto T^*$ induces a bijection between equivalence classes of tilting modules on the one hand, and equivalence classes of cotilting modules of cofinite type on the other hand.
Cofinite type

The cotilting modules and classes of the form \( T^* \) and \( \perp T^* \), respectively, are called of \textbf{cofinite type}. The map \( T \mapsto T^* \) induces a bijection between equivalence classes of tilting modules on the one hand, and equivalence classes of cotilting modules of cofinite type on the other hand.

Similarly, the maps

\[
T \mapsto (\perp T \cap \text{mod-}R)^T
\]

and

\[
C \mapsto (T^C \cap \text{mod-}R)^\perp
\]

provide for a 1–1 correspondence between tilting classes, and cotilting classes of cofinite type.
Valuation domains and cofinite type

Theorem

Let $R$ be a valuation domain. Then all cotilting classes are of cofinite type, iff $R$ is strongly discrete (that is, $R$ has no non–zero idempotent prime ideals).
Valuation domains and cofinite type

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Let $R$ be a valuation domain. Then all cotilting classes are of cofinite type, iff $R$ is strongly discrete (that is, $R$ has no non–zero idempotent prime ideals).

**Example**

Let $R$ be a maximal valuation domain with an idempotent maximal ideal $m$. Then the class of all modules $M$ whose torsion part is annihilated by $m$ is $1$–cotilting, but not of cofinite type.
The role of associated primes in the noetherian setting
A subset $P \subseteq \text{Spec}(R)$ is closed under generalization provided that $(P, \subseteq)$ is a lower subset in $(\text{Spec}(R), \subseteq)$. 
A subset $P \subseteq \text{Spec}(R)$ is **closed under generalization** provided that $(P, \subseteq)$ is a lower subset in $(\text{Spec}(R), \subseteq)$.

**Theorem (The structure of 1–cotilting classes)**
A subset \( P \subseteq \text{Spec}(R) \) is **closed under generalization** provided that \((P, \subseteq)\) is a lower subset in \((\text{Spec}(R), \subseteq)\).

**Theorem (The structure of 1–cotilting classes)**

Let \( R \) be a commutative noetherian ring. Then there is a 1–1 correspondence between

1. the 1–cotilting classes \( \mathcal{C} \) in \( \text{Mod}-R \), and
2. the subsets \( P \) of \( \text{Spec}(R) \) containing \( \text{Ass}(R) \) and closed under generalization.
A subset $P \subseteq \text{Spec}(R)$ is **closed under generalization** provided that $(P, \subseteq)$ is a lower subset in $(\text{Spec}(R), \subseteq)$.

**Theorem (The structure of 1–cotilting classes)**

Let $R$ be a commutative noetherian ring. Then there is a 1–1 correspondence between

1. the 1–cotilting classes $\mathcal{C}$ in $\text{Mod-}R$, and
2. the subsets $P$ of $\text{Spec}(R)$ containing $\text{Ass}(R)$ and closed under generalization.

It is given by the inverse assignments $\mathcal{C} \mapsto \text{Ass}(\mathcal{C})$ and $P \mapsto \{M \in \text{Mod-}R \mid \text{Ass}(M) \subseteq P\}$. 
The Auslander–Bridger transpose
Let $C \in \text{mod–}R$ and $P_1 \xrightarrow{f} P_0 \rightarrow C \rightarrow 0$ be a projective presentation of $C$. The transpose of $C$, denoted by $\text{Tr}(C)$, is the cokernel of $f^+$, where $(-)^+ = \text{Hom}_R(-, R)$. 
Let $C \in \text{mod–} R$ and $P_1 \xrightarrow{f} P_0 \rightarrow C \rightarrow 0$ be a projective presentation of $C$. The transpose of $C$, denoted by $\text{Tr}(C)$, is the cokernel of $f^+$, where $(-)^+ = \text{Hom}_R(-, R)$. That is, we have an exact sequence

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Let $C \in \text{mod–}R$ and $P_1 \xrightarrow{f} P_0 \to C \to 0$ be a projective presentation of $C$. The transpose of $C$, denoted by $\text{Tr}(C)$, is the cokernel of $f^+$, where $(-)^+ = \text{Hom}_R(-, R)$. That is, we have an exact sequence

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$\text{Tr}(C)$ is uniquely determined up to adding or splitting off a projective summand.
The Auslander–Bridger transpose

Let \( C \in \text{mod–}R \) and \( P_1 \xrightarrow{f} P_0 \rightarrow C \rightarrow 0 \) be a projective presentation of \( C \). The transpose of \( C \), denoted by \( \text{Tr}(C) \), is the cokernel of \( f^+ \), where \((-)^+ = \text{Hom}_R(-, R)\).

That is, we have an exact sequence

\[
P_0^+ \xrightarrow{f^+} P_1^+ \rightarrow \text{Tr}(C) \rightarrow 0.
\]

\( \text{Tr}(C) \) is uniquely determined up to adding or splitting off a projective summand.

**Lemma**

Let \( p \in \text{Spec}(R) \) be such that \( \text{Ass}(R) \cap V(p) = \emptyset \). Then

(i) \( pd_R(\text{Tr}(R/p)) \leq 1 \);

(ii) \( \text{Hom}_R(R/p, -) \) and \( \text{Tor}_1^R(\text{Tr}(R/p), -) \) are isomorphic functors.
A classification of 1–tilting classes
Corollary

Let $R$ be a commutative noetherian ring. Then all 1–cotilting classes are of cofinite type, so there is a bijection between 1–tilting classes and the subsets $P$ of $\text{Spec}(R)$ containing $\text{Ass}(R)$ and closed under generalization.
A classification of 1–tilting classes

Corollary

Let $R$ be a commutative noetherian ring. Then all 1–cotilting classes are of cofinite type, so there is a bijection between 1–tilting classes and the subsets $P$ of $\text{Spec}(R)$ containing $\text{Ass}(R)$ and closed under generalization. For such $P$, the corresponding 1–tilting class is

$$T = \bigcap_{q \in \text{Spec}(R) \setminus P} \text{Tr}(R/q)^\perp.$$
Characteristic sequences
Definition

Let $R$ be a commutative noetherian ring. A sequence $\mathcal{P} = (P_0, \ldots, P_{n-1})$ of subsets of $\text{Spec}(R)$ is called characteristic provided that

(i) $P_i$ is closed under generalization for all $i < n$,
(ii) $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1}$, and
(iii) $\text{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$. 

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(iii) $\text{Ass}(\Omega^{-i}(R)) \subseteq P_i$ for all $i < n$.

For each characteristic sequence $\mathcal{P}$, we define the class of modules

$$
\mathcal{C}_\mathcal{P} = \{ M \in \text{Mod-}R \mid \text{Ass}(\Omega^{-i}(M)) \subseteq P_i \text{ for all } i < n \}
$$
A classification of $n$–cotilting classes
A classification of $n$–cotilting classes

**Theorem**

Let $R$ be a commutative noetherian ring, $n \geq 1$, and $P = (P_0, \ldots, P_{n-1})$ be a characteristic sequence.
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Let $R$ be a commutative noetherian ring, $n \geq 1$, and $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_\mathcal{P}$ is an $n$–cotilting class,
A classification of $n$–cotilting classes

**Theorem**

Let $R$ be a commutative noetherian ring, $n \geq 1$, and $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence. Then $\mathcal{C}_\mathcal{P}$ is an $n$–cotilting class, and the assignments

$$\mathcal{C} \mapsto (\text{Ass}(\mathcal{C}_0), \ldots, \text{Ass}(\mathcal{C}_{n-1}))$$

and

$$\mathcal{P} = (P_0, \ldots, P_{n-1}) \mapsto \mathcal{C}_\mathcal{P}$$

are inverse bijections.
A classification of \(n\)-cotilting classes

**Theorem**

Let \(R\) be a commutative noetherian ring, \(n \geq 1\), and \(\mathcal{P} = (P_0, \ldots, P_{n-1})\) be a characteristic sequence. Then \(C_{\mathcal{P}}\) is an \(n\)-cotilting class, and the assignments

\[ C \mapsto (\text{Ass}(C_0), \ldots, \text{Ass}(C_{n-1})) \]

and

\[ \mathcal{P} = (P_0, \ldots, P_{n-1}) \mapsto C_{\mathcal{P}} \]

are inverse bijections.

**Lemma**

Let \(R\) be a ring and \(C\) be an \(n\)-cotilting module with the induced class \(C\). For each \(i \leq n\), let \(C_i = \bot \Omega^{-i}(C)\). 

Jan Trlifaj (Univerzita Karlova, Praha) 
Tilting for commutative rings
A classification of \( n \)-cotilting classes

**Theorem**

Let \( R \) be a commutative noetherian ring, \( n \geq 1 \), and \( \mathcal{P} = (P_0, \ldots, P_{n-1}) \) be a characteristic sequence. Then \( C_{\mathcal{P}} \) is an \( n \)-cotilting class, and the assignments

\[
\mathcal{C} \mapsto (\text{Ass}(C_0), \ldots, \text{Ass}(C_{n-1}))
\]

and

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\mathcal{P} = (P_0, \ldots, P_{n-1}) \mapsto C_{\mathcal{P}}
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**Lemma**

Let \( R \) be a ring and \( C \) be an \( n \)-cotilting module with the induced class \( \mathcal{C} \). For each \( i \leq n \), let \( C_i = \bot \Omega^{-i}(C) \). Then \( C_i \) is an \((n - i)\)-cotilting class.
The transpose revisited
Lemma

Let $\mathfrak{p} \in \text{Spec}(R)$ and $n \geq 1$ such that $\text{Ass}(\Omega^{-i}(R)) \cap V(\mathfrak{p}) = \emptyset$ for each $i < n$. Then

(i) $pd_R(\text{Tr}(R/\mathfrak{p})) \leq n$.

(ii) $\text{Ext}^{n-1}_R(R/\mathfrak{p}, -)$ and $\text{Tor}_1^R(\text{Tr}(\Omega^{(n-1)}(R/\mathfrak{p})), -)$ are isomorphic functors.

(iii) $\text{Ext}_R^1(\Omega^{(n-1)}(R/\mathfrak{p})), -)$ and $\text{Tor}_{n-1}^R(R/\mathfrak{p}, -)$ are isomorphic functors.
Complete classification for commutative noetherian rings
Theorem

Let $n \geq 1$. Then there are bijections between:

(i) the characteristic sequences in $\text{Spec}(R)$,
(ii) $n$–tilting classes $\mathcal{T}$,
(iii) $n$–cotilting classes $\mathcal{C}$.
Theorem

Let $n \geq 1$. Then there are bijections between:

(i) the characteristic sequences in $\text{Spec}(R)$,

(ii) $n$–tilting classes $\mathcal{T}$,

(iii) $n$–cotilting classes $\mathcal{C}$.

A characteristic sequence $(P_0, \ldots, P_{n-1})$ corresponds to the $n$–tilting class

$$\mathcal{T} = \{ M \in \text{Mod–}R \mid \text{Tor}_i^R(R/p, M) = 0 \forall i < n \forall p \notin P_i \} =$$

$$\{ M \in \text{Mod–}R \mid \text{Ext}_1^R(\text{Tr}(\Omega^{(i)}(R/p)), M) = 0 \forall i < n \forall p \notin P_i \}.$$
Theorem

Let $n \geq 1$. Then there are bijections between:

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and the $n$–cotilting class

$$\mathcal{C} = \{ M \in \text{Mod}_R | \text{Ext}_i^R(R/p, M) = 0 \forall i < n \forall p \notin P_i \} =$$

$$\{ M \in \text{Mod}_R | \text{Tor}_1^R(\text{Tr}(\Omega^i(R/p)), M) = 0 \forall i < n \forall p \notin P_i \}.$$
Minimal cotilting modules
Definition
A cotilting module $C$ is \textit{minimal} provided that $C$ is a direct summand in each cotilting module equivalent to $C$. 

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**Lemma (uniqueness)**

*If $C$ and $C'$ are minimal cotilting modules such that $C$ is equivalent to $C'$, then $C \cong C'$.***
### Definition

A cotilting module $C$ is **minimal** provided that $C$ is a direct summand in each cotilting module equivalent to $C$.

### Lemma (uniqueness)

*If $C$ and $C'$ are minimal cotilting modules such that $C$ is equivalent to $C'$, then $C \cong C'$.*

### Example

Let $R$ be a commutative noetherian ring and $C = \bigoplus_{m \in \text{Spec}(R)} E(R/m)$. Then $C$ is a minimal 0-cotilting module (= minimal injective cogenerator).
Iterated injective covers
Iterated injective covers

**Definition**

Let $R$ be commutative noetherian, and $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence. Define $P_{-1} = \emptyset$ and $P_n = \text{Spec}(R)$. 
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For each $i < n$, let $\mathcal{I}(P_i)$ be the class of all injective modules $I$ with $\text{Ass}(I) \subseteq P_i$. 
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For each $i < n$, let $\mathcal{I}(P_i)$ be the class of all injective modules $I$ with $\text{Ass}(I) \subseteq P_i$.

For each $i < n$ and each non-empty subset $S \subseteq P_i \setminus P_{i-1}$, let $E_S = \bigoplus_{p \in S} E(R/p)$ and consider the long exact sequence

$$
0 \to C_S \to E_0 \xrightarrow{\varphi_0} E_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{i-2}} E_{i-1} \xrightarrow{\varphi_{i-1}} E_S \to 0
$$
Iterated injective covers

**Definition**

Let \( R \) be commutative noetherian, and \( \mathcal{P} = (P_0, \ldots, P_{n-1}) \) be a characteristic sequence. Define \( P_{-1} = \emptyset \) and \( P_n = \text{Spec}(R) \).

For each \( i < n \), let \( \mathcal{I}(P_i) \) be the class of all injective modules \( I \) with \( \text{Ass}(I) \subseteq P_i \).

For each \( i < n \) and each non-empty subset \( S \subseteq P_i \setminus P_{i-1} \), let \( E_S = \bigoplus_{p \in S} E(R/p) \) and consider the long exact sequence

\[
0 \to C_S \to E_0 \xrightarrow{\varphi_0} E_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{i-2}} E_{i-1} \xrightarrow{\varphi_{i-1}} E_S \to 0
\]

such that \( \varphi_{i-1} \) is a \( \mathcal{I}(P_{i-1}) \)-cover of \( E_S \), and for each \( 0 < j < i - 1 \), \( \varphi_j = \mu_j \circ \psi_j \), where \( \mu_j \) is the inclusion of \( K_j = \text{Ker}(\varphi_{j+1}) \) into \( E_{j+1} \), and \( \psi_j : E_j \to K_j \) is a \( \mathcal{I}(P_j) \)-cover.
The structure of minimal cotilting modules
The structure of minimal cotilting modules

**Theorem**

Let $R$ be a commutative noetherian ring. Let $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence and $\mathcal{C}$ be the corresponding $n$-cotilting class.
The structure of minimal cotilting modules

**Theorem**

Let $R$ be a commutative noetherian ring. Let $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence and $\mathcal{C}$ be the corresponding $n$-cotilting class.

There is a minimal $n$-cotilting module $C$ inducing $\mathcal{C}$. 
The structure of minimal cotilting modules

**Theorem**

Let $R$ be a commutative noetherian ring. Let $\mathcal{P} = (P_0, \ldots, P_{n-1})$ be a characteristic sequence and $\mathcal{C}$ be the corresponding $n$-cotilting class.

There is a minimal $n$-cotilting module $C$ inducing $\mathcal{C}$.

Moreover, $C \cong C_{S_0} \oplus \cdots \oplus C_{S_n}$ where $S_i$ is the set of all maximal elements in $P_i \setminus P_{i-1}$, for all $i \leq n$. 
Cotilting and colocalization
Cotilting and colocalization

Troubles with localization of cotilting modules ...
Cotilting and colocalization

Troubles with localization of cotilting modules ...

**Definition**

Let $R$ be a commutative ring, $M$ an $R$-module, and $m \in \text{mSpec}(R)$. Denote by $M^m$ the $R_m$-module $\text{Hom}_R(R_m, M)$; it is called the colocalization of $M$ at $m$. 
Cotilting and colocalization

Troubles with localization of cotilting modules ...

**Definition**

Let $R$ be a commutative ring, $M$ an $R$-module, and $m \in m\text{Spec}(R)$. Denote by $M^m$ the $R_m$-module $\text{Hom}_R(R_m, M)$; it is called the **colocalization** of $M$ at $m$.

**Theorem**

*Let $R$ be a commutative noetherian ring, $n < \omega$, and $C$ be an $n$-cotilting $R$-module.*
Cotilting and colocalization

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Definition
Let $R$ be a commutative ring, $M$ an $R$-module, and $m \in \text{mSpec}(R)$. Denote by $M^m$ the $R_m$-module $\text{Hom}_R(R_m, M)$; it is called the colocalization of $M$ at $m$.

Theorem
Let $R$ be a commutative noetherian ring, $n < \omega$, and $C$ be an $n$-cotilting $R$-module.

Then for each $m \in \text{mSpec}(R)$, $C^m$ is an $n$-cotilting $R_m$-module, and $D = \prod_{m \in \text{mSpec}(R)} C^m$ is an $n$-cotilting $R$-module equivalent to $C$. 
Cotilting and colocalization

Troubles with localization of cotilting modules ...

**Definition**

Let $R$ be a commutative ring, $M$ an $R$-module, and $m \in \text{mSpec}(R)$. Denote by $M^m$ the $R_m$-module $\text{Hom}_R(R_m, M)$; it is called the **colocalization** of $M$ at $m$.

**Theorem**

Let $R$ be a commutative noetherian ring, $n < \omega$, and $C$ be an $n$-cotilting $R$-module. Then for each $m \in \text{mSpec}(R)$, $C^m$ is an $n$-cotilting $R_m$-module, and $D = \prod_{m \in \text{mSpec}(R)} C^m$ is an $n$-cotilting $R$-module equivalent to $C$. Moreover, $(C^m | m \in \text{mSpec}(R))$ is a **compatible family** of $n$-cotilting modules, and cotilting $R$-modules correspond 1-1 to such compatible families.
Theorem

Let $R$ be a commutative ring, $n < \omega$, and $T$ be an $n$-tilting $R$-module. Then for each $m \in \text{mSpec}(R)$, $T_m$ is an $n$-tilting $R_m$-module.
Tilting and localization

**Theorem**

Let $R$ be a commutative ring, $n < \omega$, and $T$ be an $n$-tilting $R$-module. Then for each $m \in \text{mSpec}(R)$, $T_m$ is an $n$-tilting $R_m$-module.

**Remark**

If $R$ is moreover noetherian, then $(T_m \mid m \in \text{mSpec}(R))$ is a compatible family of $n$-tilting modules. Tilting $R$-modules correspond 1-1 to such compatible families.
Tilting and localization

**Theorem**

Let $R$ be a commutative ring, $n < \omega$, and $T$ be an $n$-tilting $R$-module. Then for each $m \in \text{mSpec}(R)$, $T_m$ is an $n$-tilting $R_m$-module.

**Remark**

If $R$ is moreover noetherian, then $(T_m \mid m \in \text{mSpec}(R))$ is a compatible family of $n$-tilting modules. Tilting $R$-modules correspond 1-1 to such compatible families. However, there is no simple way to recover $T$ from the compatible family $(T_m \mid m \in \text{mSpec}(R))$. !!!!
A research outlook
1. Describe the structure of tilting modules over commutative noetherian rings.
A research outlook

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The APD and Prüfer cases are done.
References


